JOINT INSTITUTE FOR NUCLEAR RESEARCH

Bogoliubov Laboratory of Theoretical Physics

FINAL REPORT ON THE

INTEREST PROGRAMME

Numerical methods in theory of topological solitons

Supervisor:

Dr. Yakov Shnir

Student:

Marcos Alvarez Hernández, Cuba.

University of Havana

Participation period:

March 03 – April 19, Wave 12

Dubna, 2025

Contents

1 Introduction

Topological solitons are localized, stable solutions of nonlinear field equations, first studied in the context of the Fermi-Pasta-Ulam problem and subsequently elaborated into a major branch of modern theoretical physics. Such structures appear in a variety of systems from condensed matter to high-energy theory, e. g., Skyrme models of baryons. Their stability is often topologically protected; thus, they are robust against perturbations.

This project explores topological solitons in nonlinear field theories, with a focus on the O(3) sigma model, the baby Skyrme model, and the Skyrme model. Using stereographic projections, we construct explicit multi-soliton solutions and analyze their energy densities, topological charges, and stability. For the O(3) model, we derive N-soliton configurations via rational holomorphic maps. In the baby Skyrme model, we numerically investigate profile functions for $Q = 1$ solitons, comparing with known results in spontaneous symmetry breaking.Finally, we derive the spherically symmetric Skyrmion solution, compute its baryon number, and discuss its implications for holographic duality. Our work bridges analytical and numerical approaches, highlighting the role of topology in soliton stability.

2 O(3) Nonlinear Sigma Model

2.1 Stereographic Projection

The triplet of real scalar fields $\phi^a = (\phi_1, \phi_2, \phi_3)$, is restricted to the sphere S^2 via the constraint $\phi^a \cdot \phi^a = \phi_1^2 + \phi_2^2 + \phi_3^2 = 1$.

We use the stereographic projection to establish a correspondence between the points of the surface of the sphere and the points of the complex plane (parametrized by the local Euclidean coordinates (u, w)). To do so, we draw a straight line through the north pole $N(0, 0, 1)$ and any point $A\,(\phi_1, \phi_2, \phi_3)$ on the surface of the sphere. This straight line intersects the plane at some point $B(u, w, 0)$. We can write equation of our line in $\mathbb{R}^3$:

$$\frac{u}{\phi_1} = \frac{w}{\phi_2} = \frac{1}{1 - \phi_3}$$

$$(u, w) = \left(\frac{\phi_1}{1 - \phi_3}, \frac{\phi_2}{1 - \phi_3} \right) \quad (1) \tag{1}$$

The inverse transformation takes a point $B(u, w, 0)$ from the plane onto the sphere.

$$u\,(1-\phi_3)=\phi_1, \quad w\,(1-\phi_3)=\phi_2$$

Using $\phi_1^2+\phi_2^2+\phi_3^2=1$, we get:

$$(u\,(1-\phi_3))^2+(w\,(1-\phi_3))^2+\phi_3^2=1$$

$$\left(1+u^2+w^2\right)\phi_3^2-2\left(u^2+w^2\right)\phi_3+\left(u^2+w^2-1\right)=0$$

$$\phi_3=\frac{1-u^2-w^2}{1+u^2+w^2}$$

$$\phi_1=\frac{2u}{1+u^2+w^2}, \quad \phi_2=\frac{2w}{1+u^2+w^2}$$

$$(\phi_1,\phi_2,\phi_3)=\left(\frac{2u}{1+u^2+w^2},\frac{2w}{1+u^2+w^2},\frac{1-u^2-w^2}{1+u^2+w^2}\right) \quad (2) \tag{2}$$

Or, using the complex variable

$$W_N=\frac{\phi_1+i\phi_2}{1-\phi_3}=u+iw \quad (3) \tag{3}$$

$$(u,w)=\left(\frac{1}{2}\left(W_N+W_N^*\right),\frac{i}{2}\left(W_N-W_N^*\right)\right) \quad (4) \tag{4}$$

$$(\phi_1,\phi_2,\phi_3)=\left(\frac{W_N+W_N^*}{1+|W_N|^2},-i\frac{W_N-W_N^*}{1+|W_N|^2},\frac{1-|W_N|^2}{1+|W_N|^2}\right) \quad (5) \tag{5}$$

Analogously, for the south projection, we take point $S(0,0,-1)$, we get:

$$(u,w)=\left(\frac{\phi_1}{1+\phi_3},\frac{\phi_2}{1+\phi_3}\right) \quad (6) \tag{6}$$

$$(\phi_1,\phi_2,\phi_3)=\left(\frac{2u}{1+u^2+w^2},\frac{2w}{1+u^2+w^2},\frac{1-u^2-w^2}{1+u^2+w^2}\right) \quad (7) \tag{7}$$

Or, using

$$W_S = \frac{\phi_1 + i\phi_2}{1+\phi_3} = u + iw \quad (8) \tag{8}$$

we get:

$$(u, w) = \left(\frac{1}{2}\left(W_S + W_S^*\right), \frac{i}{2}\left(W_S - W_S^*\right)\right) \quad (9) \tag{9}$$

$$(\phi_1, \phi_2, \phi_3) = \left(\frac{W_S + W_S^*}{1+|W_S|^2}, -i\frac{W_S - W_S^*}{1+|W_S|^2}, \frac{1-|W_S|^2}{1+|W_S|^2}\right) \quad (10) \tag{10}$$

We can establish a relationship between (3) and (8):

$$W_S = \frac{\phi_1 + i\phi_2}{1+\phi_3} = \frac{W_N + W_N^* + W_N - W_N^*}{W_N W_N^*} = \frac{2}{W_N^*} \quad (11) \tag{11}$$

2.2 Energy Density

The energy density is given by:

$$E = \frac{1}{2}\left(\partial_\mu \phi\right)\cdot\left(\partial^\mu \phi\right) = \frac{1}{2}\left[\left(\partial_x \phi\right)^2 + \left(\partial_y \phi\right)^2\right] \quad (12) \tag{12}$$

Or, in terms of W and W^*:

$$E = \frac{|\nabla_{xy} W|^2}{\left(1+|W|^2\right)^2} = \frac{|\partial_x W|^2 + |\partial_y W|^2}{\left(1+|W|^2\right)^2} \quad (13) \tag{13}$$

Using the complex variables $z = x + iy$ and $z^* = x - iy$ we get:

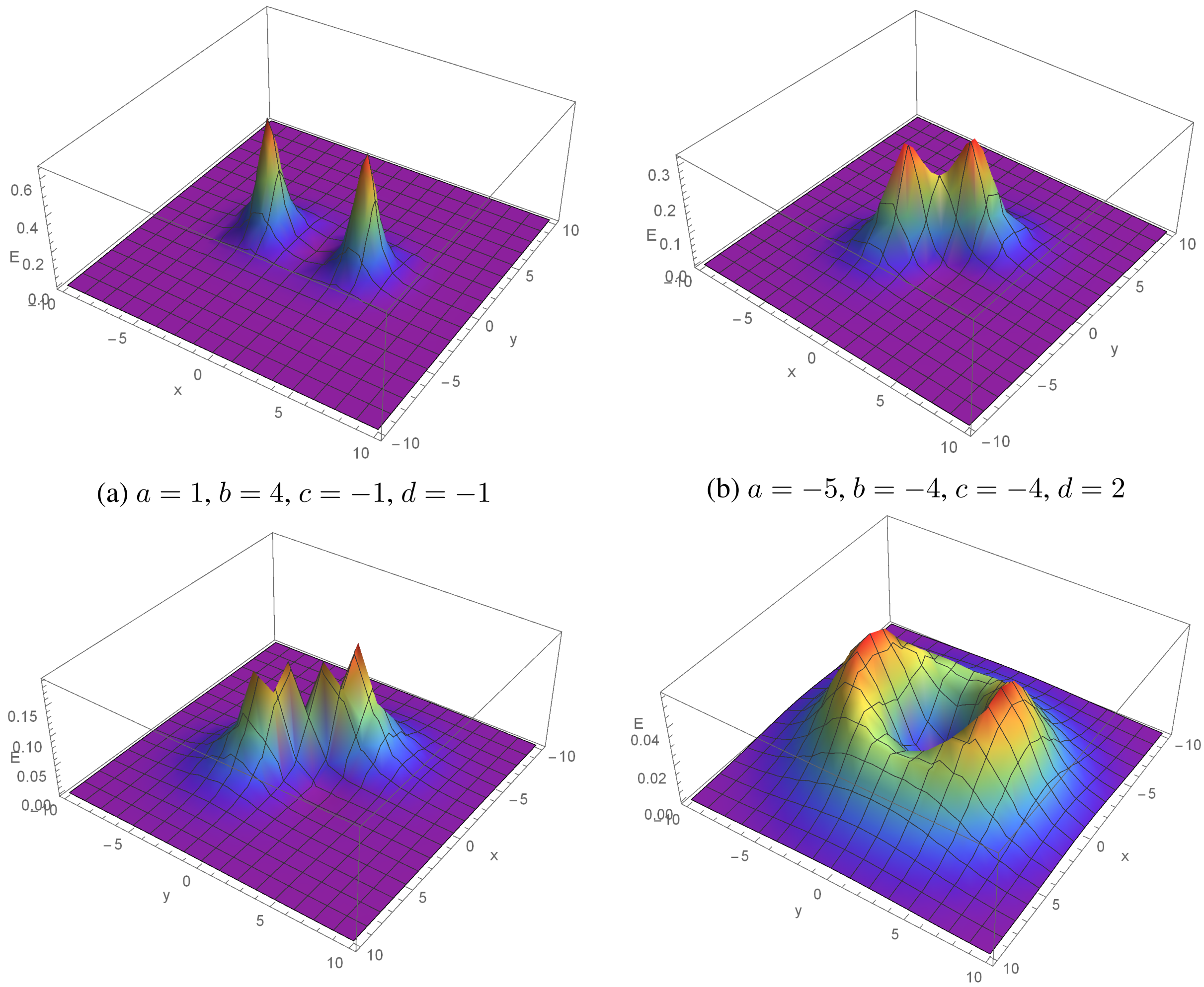
$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \partial_{z^*} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$$

$$E = \frac{|W_z|^2 + |W_{z^*}|^2}{\left(1+|W|^2\right)^2} \quad (14) \tag{14}$$

It is possible to construct any N-soliton solution of the O(3) sigma model by holomorphic map $W = P(z)/Q(z)$ where P and Q are polynomials of degree at most N of the complex coordinate $z = x + iy$. For instance, using:

$$W = \frac{(z-a)(z-c)}{(z-b)(z-d)} \quad (15) \tag{15}$$

where a, b, c, d are some complex parameters, we obtained 2-soliton solution.



(a) $a = 1, b = 4, c = -1, d = -1$

(b) $a = -5, b = -4, c = -4, d = 2$

(c) $a = -5i, b = -4, c = 5i, d = 4$

(d) $a = -5i, b = -4, c = 4i, d = 2$

Figure 1: Energy density distribution for different values of a, b, c and d

2.3 Eight Solitons Solution

We can generalize to a general N-soliton configuration. The corresponding holomorphic function W can be expressed as a rational map of degree N. For example, it has in general $4N + 2$ real parameters that fix the position of the solitons in the $\mathbb{R}^2$ plane, their shapes and sizes as well as relative orientations.

We can introduce a potential, that produces 8 solitons in the North and South projections. If in the North, the potential has the form:

$$W_N = A\left[\sum_{i=1}^{8}\frac{1}{z-z_i}\right]^{-1} \qquad (16) \tag{16}$$

And, according with (11), the south projection is given by:

$$W_S = \frac{1}{A}\sum_{n=1}^{8}\frac{1}{z^* - z_i} \quad \text{(16b)} \tag{17}$$

A chain of 8 solitons aligned on the x-axis is given by the mapping of the form

$$W_N = A\left[\sum_{n=0}^{7}\frac{1}{z-(z_0+nd)}\right]^{-1} \quad \text{(17a)} \tag{18}$$

$$W_S = \frac{1}{A}\sum_{n=0}^{7}\frac{1}{z^*-(z_0+nd)} \quad \text{(17b)} \tag{19}$$

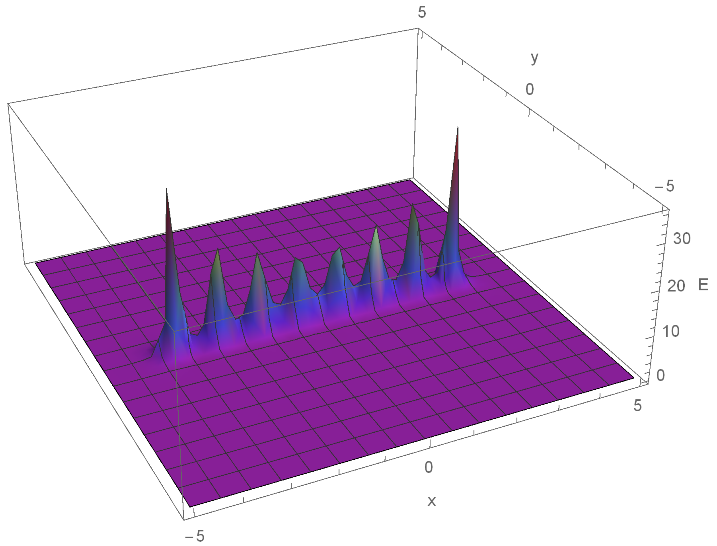


Figure 2: Chain of 8-soliton solution with $A = 4$, $z_0 = -3/5$ and $d = 1$

2.4 Eight-soliton solution for a triangle

Considering the holographic map:

$$W_N = \frac{4}{\frac{1}{z}+\frac{1}{z+\frac{1}{2}-i}+\frac{1}{z-\frac{1}{2}-i}+\frac{1}{z-1}+\frac{1}{z+1}+\frac{1}{z-\frac{3}{2}+i}+\frac{1}{z+\frac{3}{2}+i}+\frac{1}{z-2i}} \quad \text{(18a)} \tag{20}$$

An equivalent south pole projection is given by:

$$W_S = \frac{1}{4}\left(\frac{1}{z}+\frac{1}{z+\frac{1}{2}-i}+\frac{1}{z-\frac{1}{2}-i}+\frac{1}{z-1}+\frac{1}{z+1}+\frac{1}{z-\frac{3}{2}+i}+\frac{1}{z+\frac{3}{2}+i}+\frac{1}{z-2i}\right) \quad \text{(18b)} \tag{21}$$

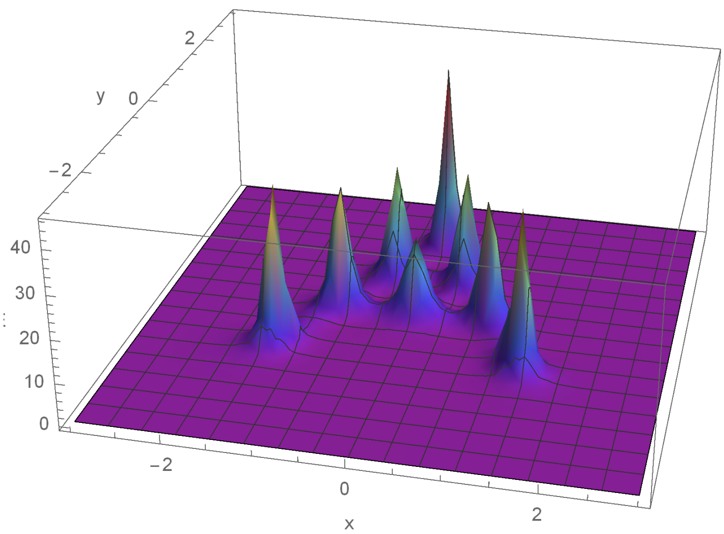
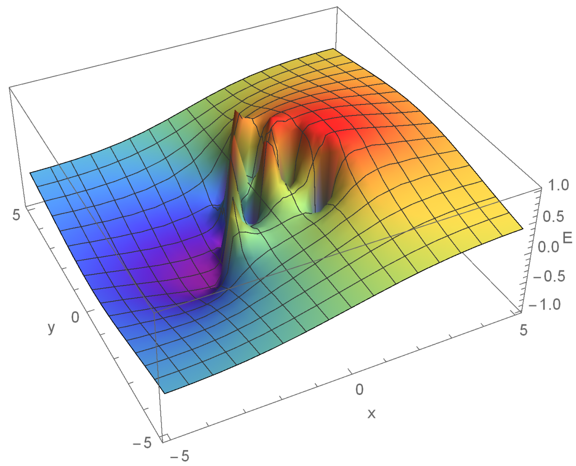
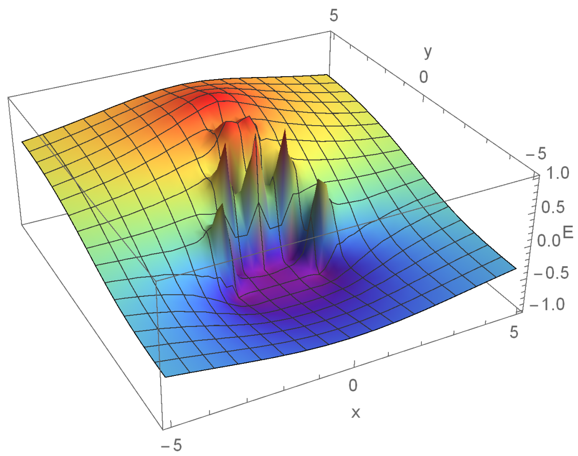


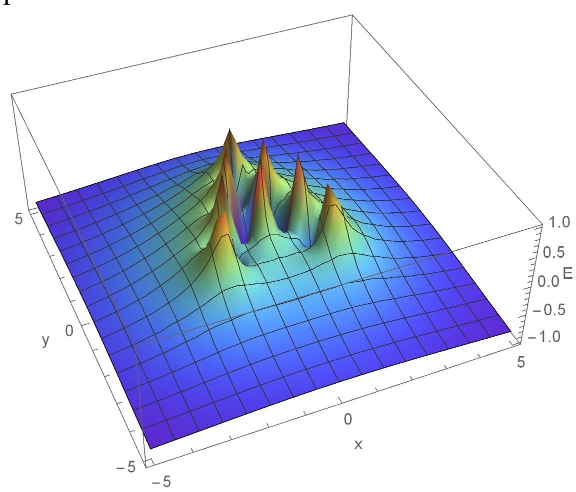
Figure 3: Energy distribution function for (17)



(a) ϕ_1 component



(b) ϕ_2 component



(c) ϕ_3 component

Figure 4: Field components of (17)

3 Skyrmions

3.1 Baby Skyrme Model

The Lagrangian of the 2+1 dimensional baby Skyrme model is:

$$L = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{1}{4}(\epsilon_{abc}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c)^2 - \mu^2(1 - \phi^3) \tag{22}$$

with the constraint:

$$\phi^a \phi^a = 1 \tag{23}$$

The ansatz (rotationally invariant configuration) is:

$$\phi^1 = \cos\theta \sin f(r), \quad \phi^2 = \sin\theta \sin f(r), \quad \phi^3 = \cos f(r) \tag{24}$$

where (r, θ) are polar coordinates and $f(r)$ is a profile function. The topological charge (or winding number) for the baby Skyrme model in 2D is given by:

$$Q = \frac{1}{4\pi} \int d^2x \, \epsilon_{ij} \epsilon^{abc} \phi^a \partial_i \phi^b \partial_j \phi^c \tag{25}$$

In terms of polar coordinates (r, θ), the Jacobian determinant for the transformation is $d^2x = rdrd\theta$. Rewriting the charge density in polar coordinates:

$$Q = \frac{1}{4\pi} \int rdrd\theta \, \epsilon_{ij} \epsilon^{abc} \phi^a \partial_i \phi^b \partial_j \phi^c$$

Using the ansatz, we compute the derivatives:

$$\partial_r \phi^1 = \cos\theta \cos f f', \quad \partial_r \phi^2 = \sin\theta \cos f f', \quad \partial_r \phi^3 = -\sin f f'$$

$$\partial_\theta \phi^1 = -\sin\theta \sin f, \quad \partial_\theta \phi^2 = \cos\theta \sin f, \quad \partial_\theta \phi^3 = 0$$

The charge density involves the determinant:

$$\epsilon^{abc} \phi^a \partial_r \phi^b \partial_\theta \phi^c$$

Computing term by term: For $a = 1, b = 3, c = 2$:

$$\epsilon^{132} = -1, \quad \phi^1 \partial_r \phi^3 \partial_\theta \phi^2 = (\cos\theta \sin f)(-\sin f f')(\cos\theta \sin f) = -\cos^2\theta \sin^2 f f',$$

for $a = 2, b = 3, c = 1$:

$$\epsilon^{231} = 1, \quad \phi^2 \partial_r \phi^3 \partial_\theta \phi^1 = (\sin\theta \sin f)(-\sin f f')(-\sin\theta \sin f) = \sin^2\theta \sin^2 f f',$$

and for other permutations:

$$\phi^1 \partial_r \phi^2 \partial_\theta \phi^3 = \phi^3 \partial_r \phi^2 \partial_\theta \phi^1 = \phi^3 \partial_r \phi^1 \partial_\theta \phi^2 = 0.$$

Adding up these terms, we get:

$$Q = \frac{1}{4\pi} \int r dr d\theta \cdot 2 \sin f f' = \frac{1}{2\pi} \int_0^\infty dr \sin f f' = \frac{1}{2}[\cos f(\infty) - \cos f(0)], \tag{26}$$

where, for $f(0) = \pi$ and $f(\infty) = 0$, we have $Q = 1$.
The energy functional for the 2+1 dimensional baby Skyrme model is:

$$E = \int d^2x \left\{ \frac{1}{2}(\partial_i \phi^a)^2 + \frac{1}{4}(\epsilon_{abc} \phi^a \partial_i \phi^b \partial_j \phi^c)^2 + U(\phi) \right\}. \tag{27}$$

The kinetic term is:

$$\frac{1}{2}(\partial_i \phi^a)^2.$$

We compute the derivatives in polar coordinates. Squaring and summing the radial derivatives over a, we get:

$$(\partial_r \phi^a)^2 = f'^2,$$

and for the angular derivatives:

$$(\partial_\theta \phi^a)^2 = \sin^2 f.$$

Since

$$\partial_i \phi^a \partial^i \phi^a = (\partial_r \phi^a)^2 + \frac{1}{r^2}(\partial_\theta \phi^a)^2,$$

we obtain:

$$\frac{1}{2}\left(f'^2 + \frac{\sin^2 f}{r^2} \right).$$

The Skyrme term is computed before and the potential term is:

$$U(\phi) = \mu^2(1 - \cos f).$$

Substitution into the energy functional yields:

$$E = 2\pi \int_0^\infty r dr \left(\frac{1}{2} f'^2 + \frac{2}{r^2} \sin^2 f (f'^2 + 1) + \mu^2(1 - \cos f) \right). \tag{28}$$

Variation of this functional with respect to f yields the following equation:

$$\frac{d}{dr}\left[(1 - \frac{\sin^2 f}{r^2}) f'\right] - \left[\frac{2\sin f \cos f}{r^2} - \frac{\sin f f'^2}{r} - \mu^2 \sin f\right] = 0$$

$$(1 - \frac{\sin^2 f}{r^2}) f'' - \frac{2\sin f \cos f}{r^2} f'^2 + \frac{\sin f \cos f}{r^2} - \frac{\sin f f'^2}{r} - \mu^2 \sin f = 0$$

$$\left(r + \frac{\sin^2 f}{r}\right) f'' + \left(1 - \frac{\sin^2 f}{r^2} + \frac{f' \sin f \cos f}{r}\right) f' - \frac{\sin f \cos f}{r} - r\mu^2 \sin f = 0 \tag{29}$$

A graphic of $f(v)$ vs v for several values of μ is shown in [Figure 1].

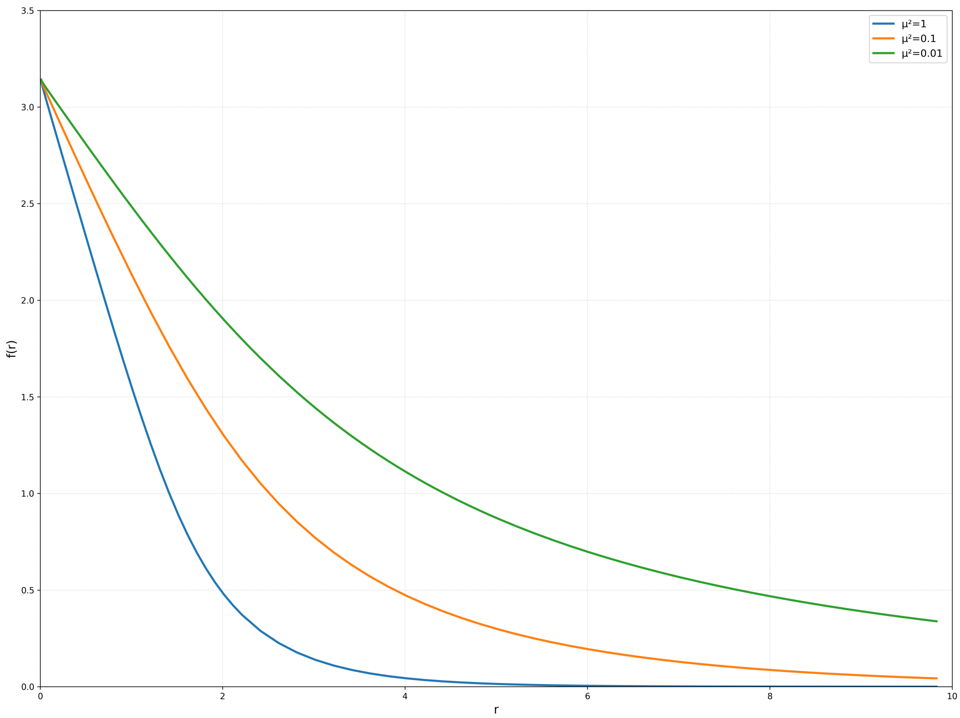


Figure 5: Profile functions $f(r)$ of the $Q = 1$ baby skyrmions for various μ^2.

3.2 Skyrme Model

The Skyrme model is a (3+1)-dimensional field theory where the field is a mapping $U : \mathbb{R}^3 \to$ $\mathrm{SU}(2)$, described by the Lagrangian:

$$L = -\frac{1}{2}\mathrm{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial^\mu U)] + \frac{1}{16}\mathrm{tr}[(\partial_\mu U)U^\dagger, (\partial_\mu U)U^\dagger]^2. \tag{30}$$

The topological charge or baryon number is given by:

$$Q = \frac{1}{24\pi^2}\int d^3x\ \epsilon^{ijk}\mathrm{tr}\left((U^\dagger \partial_i U)(U^\dagger \partial_j U)(U^\dagger \partial_k U)\right). \tag{31}$$

A static, spherically symmetric ansatz for the Skyrmion is:

$$U(r) = e^{if(r)\hat{r}\cdot\tau} = \cos f(r) + i\sin f(r)(\hat{r}\cdot\tau), \tag{32}$$

where $\hat{r}\cdot\tau = \tau^i \frac{x^i}{r}$ represents the orientation in the SU(2) group space. Taking derivatives: Differentiating:

$$\partial_i U = (-\sin f f')\hat{r}^a\tau^a + i\cos f f'\hat{r}^a\tau^a + i\sin f \partial_i(\hat{r}^a\tau^a). \tag{33}$$

We need to compute $\partial_i(\hat{r}^a\tau^a)$. Since:

$$\hat{r}^a = \frac{x^a}{r},$$

we get:

$$\partial_i \hat{r}^a = \frac{\delta_{ia} r - x^a \partial_i r}{r^2} = \frac{\delta_{ia} - \hat{r}^a\hat{r}_i}{r}.$$

Thus:

$$\partial_i(\hat{r}^a\tau^a) = \frac{\delta_{ia} - \hat{r}^a\hat{r}_i}{r}\tau^a + \epsilon_{iab}\frac{\hat{r}^b}{r}\tau^b.$$

Substituting this into $\partial_i U$:

$$\partial_i U = i\tau^a\left(\hat{r}^a\hat{r}_i f' + \frac{\delta_{ia} - \hat{r}^a\hat{r}_i}{r}\sin f\cos f + \epsilon_{iab}\frac{\hat{r}^b}{r}\sin^2 f\right).$$

Since:

$$U^\dagger = \cos f - i\sin f(\hat{r}^a\tau^a),$$

multiplying,

$$L_i = U^\dagger \partial_i U = i\tau^a \left(\hat{r}^a \hat{r}_i f' + \frac{\delta_{ia} - \hat{r}^a \hat{r}_i}{r} \sin f \cos f + \epsilon_{iab} \frac{\hat{r}^b}{r} \sin^2 f \right) = i\tau^a l_{ai}, \tag{34}$$

where:

$$l_{ai} = \hat{r}^a \hat{r}_i f' + \frac{\delta_{ia} - \hat{r}^a \hat{r}_i}{r} \sin f \cos f + \varepsilon_{iab} \frac{\hat{r}^b}{r} \sin^2 f. \tag{35}$$

We use the identity:

$$\mathrm{Tr}(\tau^a \tau^b \tau^c) = 2i\epsilon^{abc}.$$

This allows us to simplify the expression for Q:

$$Q = -\frac{1}{24\pi^2} \int 2i\epsilon^{ijk} \epsilon^{abc} l_{ai} l_{bj} l_{ck} \, d^3x.$$

Using the Levi-Civita contraction:

$$\epsilon^{ijk} \epsilon^{abc} = 6(\delta^{ia} \delta^{jb} \delta^{kc} + \text{permutations}),$$

we obtain:

$$\begin{aligned} Q &= -\frac{1}{12\pi^2} \int 6 \left[f' \left(\frac{\sin 2f}{2r} \right)^2 + f' \frac{\sin^4 f}{r^2} \right] r^2 dr d\Omega \\ &= -\frac{1}{2\pi^2} \int \left[f' \frac{4\sin^2 f \cos^2 f}{4r^2} + f' \frac{\sin^2 f \sin^2 f}{r^2} \right] r^2 dr d\Omega \\ &= -\frac{1}{2\pi^2} \int f' \frac{\sin^2 f}{r^2} r^2 dr d\Omega = -\frac{2}{\pi} \int_0^\infty f' \sin^2 f \, dr \\ &= -\frac{1}{\pi} \int_{f(0)}^{f(\infty)} (1 - \cos 2f) \, df = \frac{1}{\pi} \left[f - \frac{\sin 2f}{2} \right]_{f(0)}^{f(\infty)}. \end{aligned} \tag{36}$$

If we impose the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$, we get $Q = 1$, which corresponds to the spherically symmetric unit charge skyrmion. Setting the boundary conditions $f(0) = -\pi$ and $f(\infty) = 0$, then $Q = -1$, the anti-skyrmion solution.

The Skyrme model is based on a field $U(x) \in \mathrm{SU}(2)$ and has the Lagrangian:

$$L = \frac{F_\pi^2}{16} \mathrm{Tr} \left(\partial_\mu U^\dagger \partial^\mu U \right) + \frac{1}{32e^2} \mathrm{Tr} \left([U^\dagger \partial_\mu U, U^\dagger \partial_\nu U]^2 \right) + \frac{m^2}{8} \mathrm{Tr}(U - I). \tag{37}$$

The energy functional is given as:

$$E = -\int d^3x \left\{ \frac{1}{2}\text{Tr}[L_i L_i] + \frac{1}{16}\text{Tr}[L_i, L_j]^2 + m^2\text{Tr}(U - \mathbb{I}) \right\}. \tag{38}$$

Using $L_i = i\tau^a l_{ai}$, we compute:

$$\text{Tr}[L_i L_i] = \text{Tr}[(-\tau^a l_{ai})(-\tau^b l_{bi})].$$

In order to compute the first term, we use the trace identity for Pauli matrices:

$$\text{Tr}(\tau^a \tau^b) = 2\delta^{ab},$$

we get:

$$\text{Tr}[L_i L_i] = 2 l_{ai} l_{ai}.$$

Now, computing $l_{ai} l_{ai}$:

$$l_{ai} l_{ai} = f'^2 + \frac{2\sin^2 f}{r^2}.$$

$$\frac{1}{2}\text{Tr}[L_i L_i] = f'^2 + \frac{2\sin^2 f}{r^2}.$$

For the Skyrme term, the commutator is:

$$[L_i, L_j] = i\varepsilon^{abc}\tau^c l_{ai} l_{bj}.$$

Taking the trace:

$$\text{Tr}[L_i, L_j]^2 = 2(2\sin^2 f f'^2 + \frac{\sin^4 f}{r^2}).$$

So,

$$\frac{1}{16}\text{Tr}[L_i, L_j]^2 = \frac{1}{8}(2\sin^2 f f'^2 + \frac{\sin^4 f}{r^2}) = \frac{1}{4}\sin^2 f f'^2 + \frac{1}{8}\frac{\sin^4 f}{r^2}.$$

And for the mass term:

$$m^2\text{Tr}(U - \mathbb{I}) = 2m^2(1 - \cos f).$$

Finally, we get:

$$E = 4\pi \int_0^\infty dr \left(r^2 f'^2 + 2\sin^2 f(1 + f'^2) + \frac{\sin^4 f}{r^2} + m^2(1 - \cos f) \right). \tag{39}$$

The variation of this functional yields the ordinary differential equation of second order:

$$(r^2 + 2\sin^2 f)f'' + 2rf' - \sin 2f(1 - f'^2 + \frac{\sin^2 f}{r^2}) + m^2 \sin f = 0. \tag{40}$$

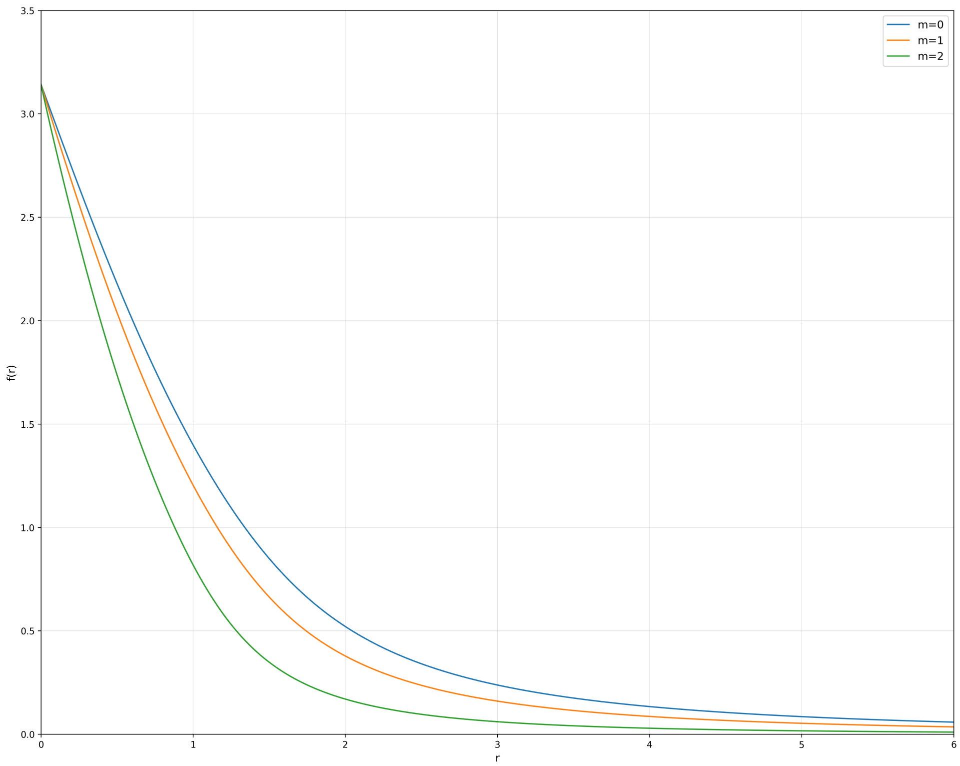


Figure 6: Figure 2: Plot of $f(r)$ for the Skyrme model for different values of m.

4 Conclusions

Using stereographic projections (both north and south pole), we successfully constructed and visualized the energy density distributions for 2-soliton and 8-soliton solutions. The field components were shown to be independent of the projection choice, confirming the geometric consistency of the approach. The solutions were generalized to N-soliton configurations via rational holomorphic maps $W = P(z)/Q(z)$, demonstrating the flexibility of this method for arbitrary soliton numbers.

The profile function $f(r)$ for the $Q = 1$ soliton was solved numerically, with results matching theoretical expectations. As predicted, increasing the rescaled mass parameter μ^2 led to a faster decay of $f(r)$, reflecting the stronger influence of the potential term. The spherically symmetric Skyrmion solution was derived, and its baryon number was verified through direct computation of the topological charge. The energy density profile highlighted the role of the Skyrme term in stabilizing the soliton against scaling collapse.

These results underscore the power of combining analytical methods like stereographic projections with numerical tools to study topological solitons across dimensions.

References

[1] Y. M. Shnir, *Topological SolitonsTopological and non-topological solitons in scalar field theories.* Cambridge University Press, 2018, ISBN 978-1-108-63625-4