Numerical Methods is the Theory of Topological Solitons

JINR INTEREST Report

submitted by

Aleksey Mikhalyuk

supervised by

Prof. Yakov Shnir

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Contents

1 Introduction

Due to Derrick's theorem in 2-dimensional space, scalar fields for soliton solutions must always be in a vacuum state. Nevertheless, by choosing certain boundary conditions of the vacuum, we can obtain topologically nontrivial soliton solutions. Such solutions take place in various sigma models.[1]

Sigma models are presented, when considering situations, where the symmetry of the initial system **G** is violated to some symmetry **H**, which acts trivially on vacuum fields. It is assumed that all vacuum fields can be obtained transitively by the action of the symmetry group of the initial system on some vacuum ϕ_0. Then the manifold **M** corresponding to the set of vacuum values is associated with the group of the initial system as follows[2]:

$$M = G/H = \{gH : g \in G\}$$

An example is the violation of the chiral symmetry $SU(3) \times SU(3)$ to the vector $SU(3)$. In this report, a fairly simple sigma $O(3)$ model will be considered. If the $SO(3)$ group is violated to $SO(2)$, then the vacuum fields take values on:

$$S^2 = SO(3)/SO(2)$$

The Lagrangian of the sigma model must be invariant with respect to the action of the transitive group G, then the main part will have the form :

$$\mathcal{L} = \frac{f}{2} g_{ij} \partial^\mu \theta^i \partial_\mu \theta^j \tag{1}$$

Where f is some constant, θ are coordinates on M , $g_{ij} = g_{ij}(\theta)$- metric on M, which is invariant with respect to the action of the group G. In our model $\phi^a\phi^a$=1, {a = 1, 2, 3}-field triplet, and our Lagrangian has the form:

$$\mathcal{L} = \frac{1}{4} \partial^\mu \phi^a \partial_\mu \phi^a \tag{2}$$

The soliton solution of our model can be constructed in terms of the complex variable W using the north (or south) stereographic projection from the sphere S^2 onto the plane $\mathbb{R}^2$.

2 Stereographic projection

The triplet takes values on a S^2:

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 1 \tag{3}$$

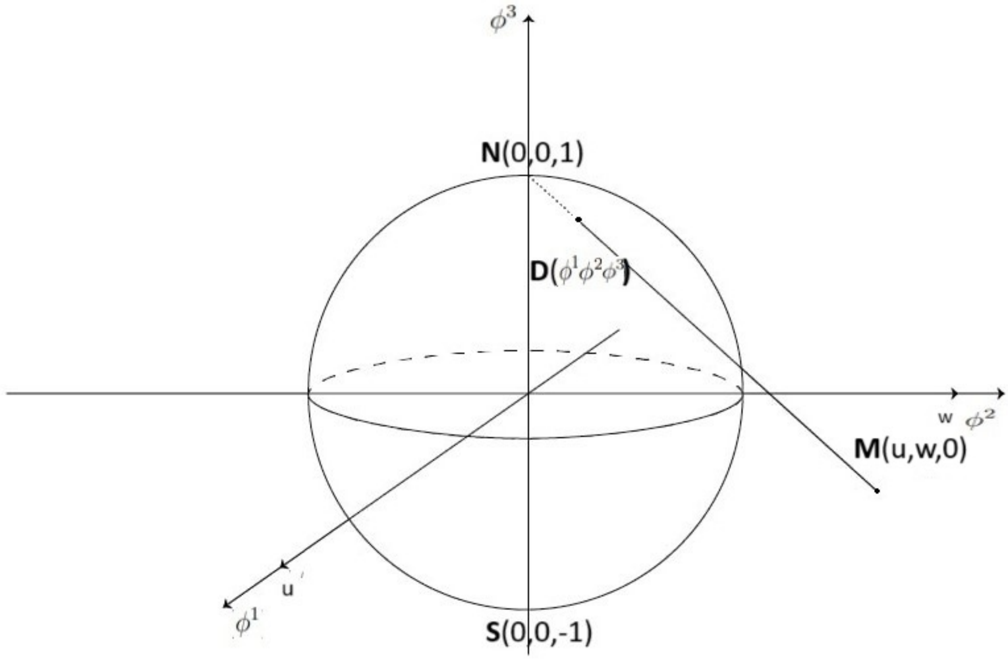


Figure 1: Stereographic projection

We'll try to find a one-to-one map from S^2 to $\mathbb{R}^2$.

Let's draw a straight line through point $N(0,0,1)$ (**North projection**) and any point on the sphere $D(\phi^1,\phi^2,\phi^3)$. This straight line intersects our plane at some point $M(u,w,0)$. We can write equation of our line in $\mathbb{R}^3$:

$$\frac{x-0}{\phi^1-0}=\frac{y-0}{\phi^2-0}=\frac{z-1}{\phi^3-1}$$

So, now we can put coordinates of M(u,w,0) in our equation:

$$\frac{u}{\phi^1}=\frac{w}{\phi^2}=\frac{1}{1-\phi^3} \tag{4}$$

Was obtained the following mapping of the points of the sphere onto the plane:

$$(u,w)=(\frac{\phi^1}{1-\phi^3},\frac{\phi^2}{1-\phi^3}) \tag{5}$$

From equations (3) and (4) we can get inverse mapping:

$$(\phi^1, \phi^2, \phi^3) = (\frac{2u}{1+u^2+w^2}, \frac{2w}{1+u^2+w^2}, -\frac{1-u^2-w^2}{1+u^2+w^2}) \tag{6}$$

If we take point $S(0,0,-1)$ (**South projection**) instead of $N(0,0,1)$, we will get such mapping (in equations (5) and (6) we just change sign):

$$(u', w') = (\frac{\phi^1}{1+\phi^3}, \frac{\phi^2}{1+\phi^3}) \tag{7}$$

And inverse:

$$(\phi^1, \phi^2, \phi^3) = (\frac{2u'}{1+u'^2+w'^2}, \frac{2w'}{1+u'^2+w'^2}, \frac{1-u'^2-w'^2}{1+u'^2+w'^2}) \tag{8}$$

Let's see how our metric is transformed using this mapping. In $\mathbb{R}^3$,the interval $ds^2 = d\phi^a d\phi_a = g_{ab} d\phi^a d\phi^b = (d\phi^1)^2 + (d\phi^2)^2 + (d\phi^3)^2$, where $g_{ab} = diag(1,1,1)$, $a, b = 1..3$. Then we use the equation (6) (or (8), as you can see $(d\phi^3)^2$ doesn't depend on the choice of projection, so our Lagrangian is independent too) and after simple transformations we get:

$$d\phi^a d\phi^a = \frac{4}{(1+u^2+w^2)^2}(du^2 + dw^2)$$

Such form of metric $g = g(u, w)$ is called *conformal*, and coordinates u, w in which metric has a conformal form called *conformal coordinates*[3]. And now we can rewrite Lagrangian (2):

$$\mathcal{L} = \frac{1}{(1+u^2+w^2)^2}((\partial^a u)^2 + (\partial^a w)^2) \tag{9}$$

We can use such a variable substitution:

$$\begin{cases} W = u + iw \\ \overline{\mathrm{W}} = u - iw \end{cases} \Longrightarrow \begin{cases} u = \frac{1}{2}(W + \overline{\mathrm{W}}) \\ w = \frac{i}{2}(W - \overline{\mathrm{W}}) \end{cases} \tag{10}$$

So, Lagrangian will have the form:

$$\mathcal{L} = \frac{\partial^a W \partial_a \overline{\mathrm{W}}}{(1 + W\overline{\mathrm{W}})^2} \tag{11}$$

If new derivatives are introduced ∂_z and $\partial_{\overline{z}}$, we'll get:

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}) \\ \partial_{\overline{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}) \end{cases} \Longrightarrow \mathcal{L} = \frac{|\partial_z W|^2 + |\partial_z \overline{\mathrm{W}}|^2}{(1 + W\overline{\mathrm{W}})^2} \tag{12}$$

From (6) (North projection) and (10) we can write $\phi^a = \phi^a(W, \overline{\mathrm{W}})$ and $W = W(\phi^a), a = 1..3$:

$$\begin{cases} \phi^1 = \frac{W+\overline{\mathrm{W}}}{1+|W|^2} \\ \phi^2 = -i\frac{W-\overline{\mathrm{W}}}{1+|W|^2} \\ \phi^3 = -\frac{1-|W|^2}{1+|W|^2} \end{cases} \Longrightarrow W^N = \frac{\phi^1 + i\phi 2}{1-\phi^3} \tag{13}$$

From (8) (South projection) and (10):

$$\begin{cases} \phi^1 = \frac{W+\overline{\mathrm{W}}}{1+|W|^2} \\ \phi^2 = -i\frac{W-\overline{\mathrm{W}}}{1+|W|^2} \\ \phi^3 = \frac{1-|W|^2}{1+|W|^2} \end{cases} \Longrightarrow W^S = \frac{\phi^1 + i\phi 2}{1+\phi^3} \tag{14}$$

3 Energy density

Let's find stress-energy tensor and energy density of Lagrangian (1):

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \partial^\nu \phi^a - \eta^{\mu\nu} \mathcal{L}$$

$$E = T^{00} = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi^a} \partial^0 \phi^a - \eta^{00} \mathcal{L} = \mathcal{L} + 2(\partial_0 \phi^a)^2 \tag{15}$$

In stationary model energy density is equal to Lagrangian and after coordinate transformation have the from (12)

4 Two soliton solution

Now, can be presentated simple soliton solution (in this report, the North projection will be used). We construct the soliton solution of the O(3) sigma model via holomorphic map:

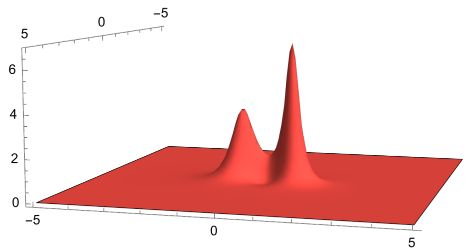
$$W = \frac{P(z)}{Q(z)}, \tag{16}$$

where P(z),Q(z)- polynomials of degree at most N of the complex coordinate $z = x_1 + ix_2$/par

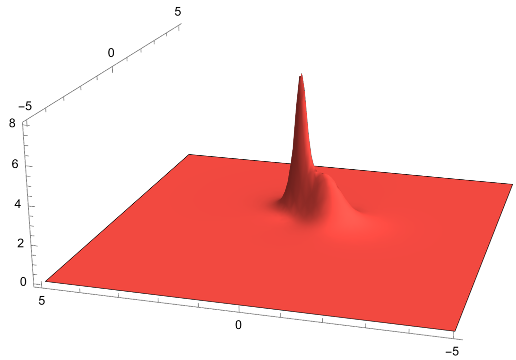
To get two soliton solution we will use such W=W(z):

$$W = \frac{(z-a)(z-b)}{(z-c)(z-d)}, \tag{17}$$

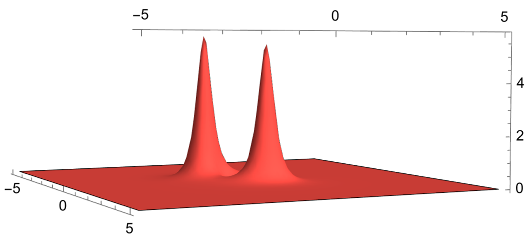
where a,b,c,d- some complex numbers.



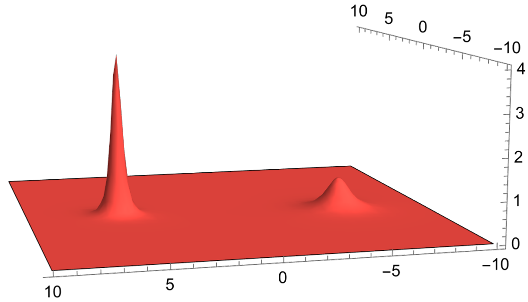
(a) Energy density distribution for
$a = -0.08 - i, b = 0.24 + i,$
$c = 0.8 + 0.27i, d = 1 - 0.28i$



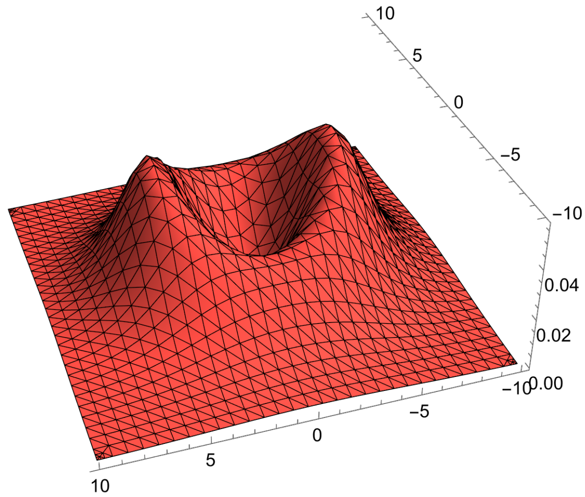
(b) Energy density distribution for
$a = -0.006 - i, b = 1 + 0.5i,$
$c = 2 - 0.27i, d = 1 - 0.28i$



(c) Energy density distribution for
$a = -1 - 1i, b = 1 + 0.5i,$
$c = 0.002 - 0.27i, d = 0.006 - 0.28i$



(d) Energy density distribution for
$a = -2 - 6i, b = 0 + 6i,$
$c = -5i, d = 5i$



(e) Energy density distribution for
$a = -0 - 6i, b = 0 + 6i,$
$c = -5, d = 5$

Figure 2: Energy distribution for different values of a,b,c,d

5 Eight solitons solution

Using equations (13) and (14), we can find how W^N and W^S corresponds to each other:

$$W^S = \frac{\phi^1 + i\phi 2}{1+\phi^3} = \frac{W^N + \overline{\mathrm{W^N}} - \overline{\mathrm{W^N}} + W^N}{W^N \overline{\mathrm{W^N}}} = \frac{2}{\overline{\overline{\mathrm{W^N}}}} \tag{18}$$

The constant multiplier can be neglect.

We can introduce a potential, that produces 8 solitons in a row in the North and South projections. If in the North, the potential has the form:

$$W^N = \frac{\alpha}{\frac{1}{z-x_1}+\frac{1}{z-x_2}+\frac{1}{z-x_3}+\frac{1}{z-x_4}+\frac{1}{z-x_5}+\frac{1}{z-x_6}+\frac{1}{z-x_7}+\frac{1}{z-x_8}} \tag{19}$$

Then in the South:

$$W^S = \frac{1}{\alpha}\left(\frac{1}{\overline{\mathrm{z}}-x_1}+\frac{1}{\overline{\mathrm{z}}-x_2}+\frac{1}{\overline{\mathrm{z}}-x_3}+\frac{1}{\overline{\mathrm{z}}-x_4}+\frac{1}{\overline{\mathrm{z}}-x_5}+\frac{1}{\overline{\mathrm{z}}-x_6}+\frac{1}{\overline{\mathrm{z}}-x_7}+\frac{1}{\overline{\mathrm{z}}-x_8}\right)$$

Where α is scale coefficient, for constructing solitons was used $\alpha = 10$, x_i-position of solitons on the x-axis, i=1..8.

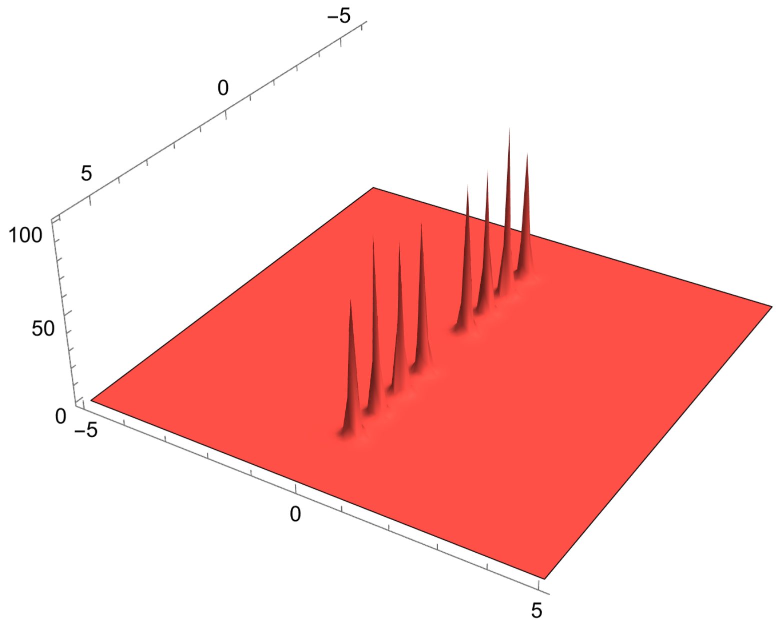


Figure 3: Eight solitons solution for
$x_1 = -3.5, x_2 = -2.5, x_3 = -1.5, x_4 = -0.5, x_5 = 0.5, x_6 = 1.5,$
$x_7 = 2.5, x_8 = 3.5$

6 Eight solitons solution in a triangle

As was introduced in previous section $W = W(z)$, which creates eight solitons solution in raw, now can be introduced a holomorphic map that presents eight solitons solution in a triangle:

$$W^N = \frac{4}{\frac{1}{z} + \frac{1}{z+\frac{1}{2}-i} + \frac{1}{z-\frac{1}{2}-i} + \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z+\frac{3}{2}+i} + \frac{1}{z-\frac{3}{2}+i} + \frac{1}{z-2i}} \tag{20}$$

$$W^S = \frac{1}{4}\left(\frac{1}{\overline{z}} + \frac{1}{\overline{z}+\frac{1}{2}+i} + \frac{1}{\overline{z}-\frac{1}{2}+i} + \frac{1}{\overline{z}-1} + \frac{1}{\overline{z}+1} + \frac{1}{\overline{z}+\frac{3}{2}-i} + \frac{1}{\overline{z}-\frac{3}{2}-i} + \frac{1}{\overline{z}+2i}\right)$$

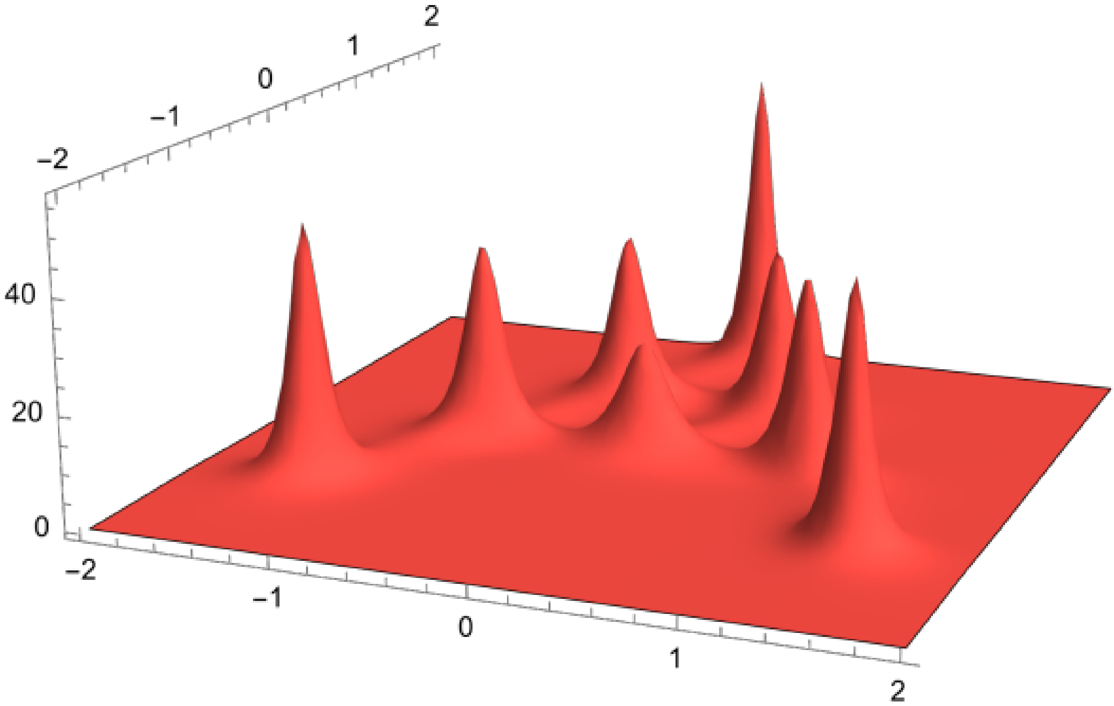
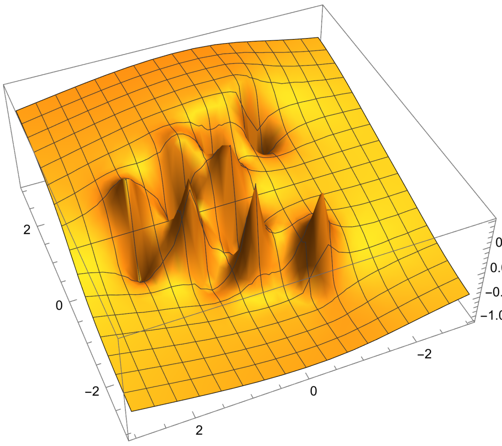
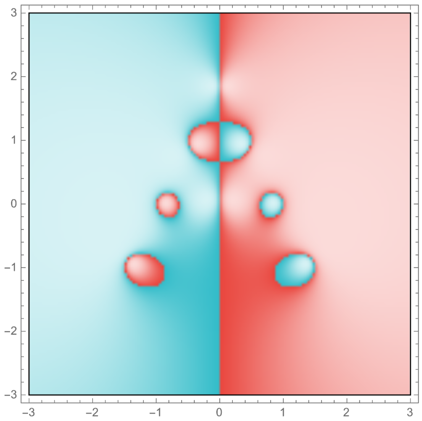


Figure 4: Eight solitons solution in a triangle

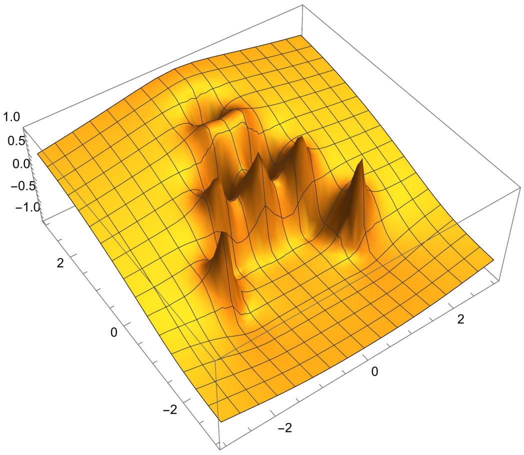
Using inverse mapping (13), can be found ϕ^1, ϕ^2, ϕ^3:

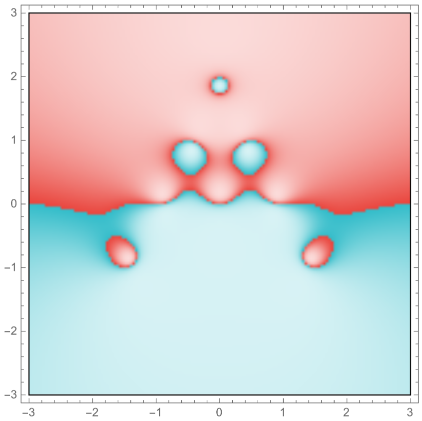




(a) ϕ^1 potential

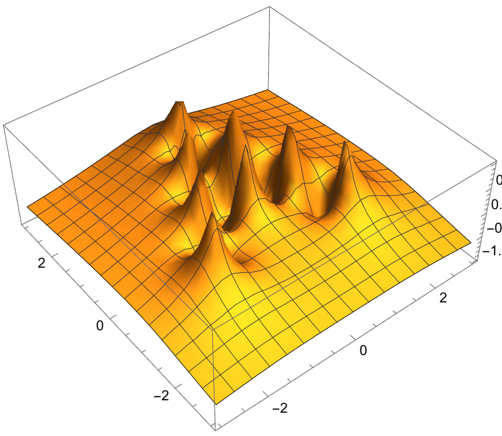
(b) ϕ^1 potential

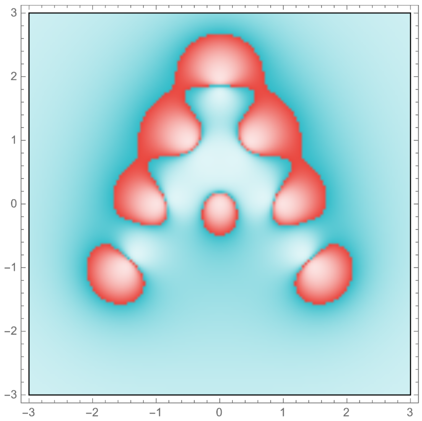




(c) ϕ^2 potential

(d) ϕ^2 potential





(e) ϕ^3 potential

(f) ϕ^3 potential

Figure 5: Potentials ϕ^1, ϕ^2, ϕ^3

7 Q-Lumps in the $O(3)$ Sigma Model

In this section will be discused non-linear $O(3)$ Sigma Model with sich Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi_i - V(\phi^i), \tag{21}$$

where $i = 1, 2, 3$; $\phi^i \phi_i = 1$ and $V(\phi^i)$ is a potential of the scalar field.

Derrick's theorem excludes existence of a localized static soliton solution for such system, but this can be circumvented by intoducing isorotations of soliton configuration[4].

Introduce potential $V(\phi^3) = \frac{\mu^2}{4}(1 - (\phi^3)^4)$. The minimum value corresponding to this potential is achieved for $\phi^3 = 1, \phi^1 = \phi^2 = 0$. So our $O(3)$ symmetry system brokes to O(2) around our vacuum value field. So, our system is invariant with respect to the isorotations:

$$\phi^1 \longmapsto \phi^1 \cos(\omega t) + \phi^2 \sin(\omega t),$$

$$\phi^2 \longmapsto -\phi^2 \cos(\omega t) + \phi^1 \sin(\omega t),$$

Corresponding Noether current:

$$j_m u = \varepsilon_{ab} \partial_m u \phi^a \phi^b$$

Then the conserved charge of our configuration:

$$Q_{SO(2)} = \int j_0 \, dx^2 = \omega \int (\phi^2)^2 + (\phi^1)^2 \, dx^2 \tag{22}$$

As was admitted previously, because of $|\phi|^2 = 1$ soliton solutions will be topologycally-nontrivial. Our scalar fields lives on S^2_ϕ, and all fields approach on vacuum field at infinity, so all the points on boundary are identical and as a result the coordinate plane $\mathbb{R}^2 \mapsto S^2$. So the field becomes a map $\phi : S^2 \mapsto S^2$. As a result, we have topological charge Q, the number of times the sphere S^2 wrapped around the sphere S^2_ϕ.

$$Q = \frac{1}{8\pi} \int \varepsilon_{abc} \varepsilon_{ij} \phi^a \partial_i \phi^b \partial_j \phi^c.$$

Configuration which carries topological and Noether charge was called *Q-lumps.* Our energy functional has the form:

$$E[\phi] = \int \frac{1}{2}\partial_j \phi^i \partial^j \phi_i + V(|\phi|) + \frac{\omega^2}{2}((\phi^2)^2 + (\phi^1)^2) \, dx^d,$$

$$E[\phi] = E_T + E_V + \frac{Q^2_{SO(2)}}{2\Lambda},$$

where $\Lambda = \int (\phi^2)^2 + (\phi^1)^2)\, dx^d$- the moment of inertia.

After scaling transformation $x \mapsto \lambda x$ and finding of the stationary point of the energy E at $\lambda = 1$, we will get:

$$(2-d)E_T + d(\frac{Q^2_{SO(2)}}{2\Lambda} - E_V) = 0$$

In our model $d = 2$, so

$$E_V = \frac{Q^2_{SO(2)}}{2\Lambda} \tag{23}$$

Stable configurations will exist only for some values of ω. From (23) we can find boundaries of changing for ω[6],[4]:

$$\frac{1}{\sqrt{2}} \leq \omega \leq m \tag{24}$$

Now we can consider two variational problems for finding corresponding field equations:

$$\begin{aligned} E_{Q_{SO(2)}}[\phi] &= E_T + E_V + \tfrac{Q^2_{SO(2)}}{2\Lambda}, \\ F_\omega[\phi] &= E_T + E_V - \tfrac{\Lambda\omega^2}{2}, \end{aligned} \tag{25}$$

the energy functional extremized with fixed $Q_{SO(2)}$, the pseudoenergy functional extremized with fixed ω. There is some difficulties related with solving differential-integral equation for the first functional, because Λ is in the denominator [4]. Therefore, it is preferable to solve the equations obtained from the second functional.

The isorotations of the configurarion (25) don't violate the symmetry of the system, so we can use hedgehog ansatz:

$$\begin{aligned} \phi^1 &= \cos\psi \sin f(r), \\ \phi^2 &= \sin\psi \sin f(r), \\ \phi^3 &= \cos f(r), \end{aligned} \tag{26}$$

where f(r) some radial function, with boudary conditions: $f(0) = \pi, f(\text{inf}) = 0$. After substitution of this parameterization and solving the variatonal problem for pseudoenergy (25), we will get following equation:

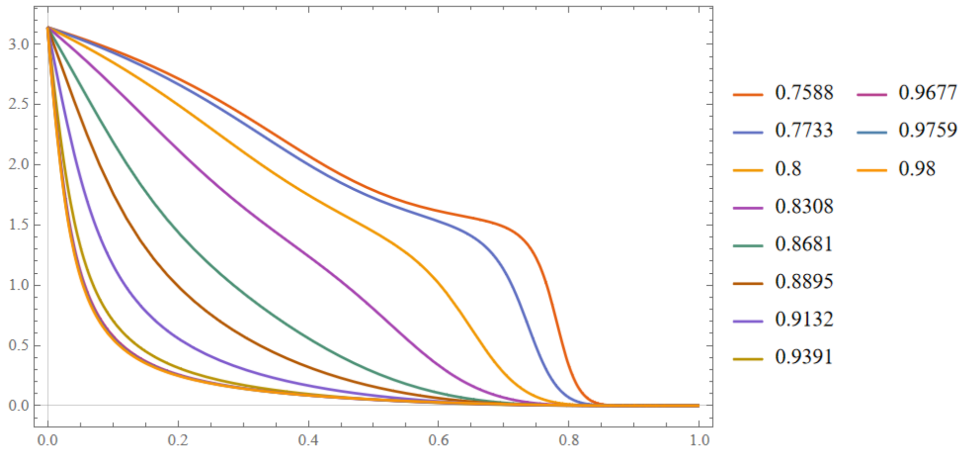
$$f'' + \frac{1}{r}f' + (\omega^2 - \frac{1}{r^2})\sin f \cos f - m^2 \cos^3 f \sin f = 0. \tag{27}$$

And corresponding charges:

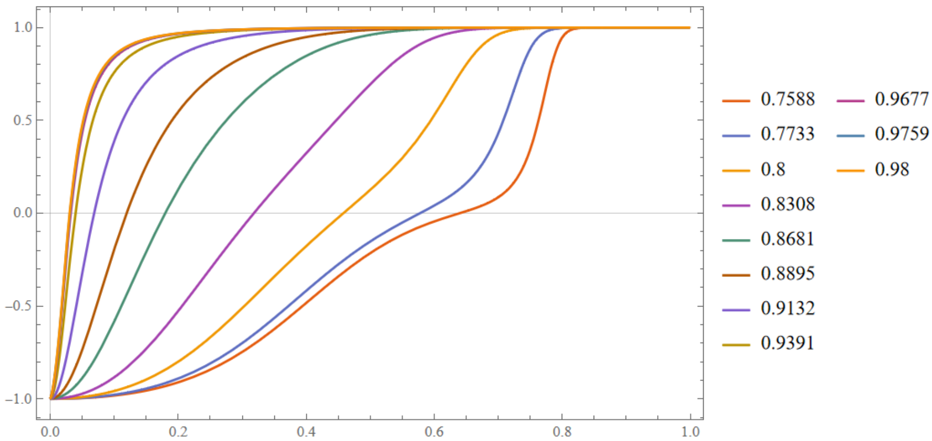
$$Q_{SO(2)} = 2\pi\omega \int_0^\infty r \sin^2 f \, dr,$$

$$Q = -\frac{1}{2} \int_0^\infty f' \sin f \, dr = \frac{1}{2}(1 - \cos f(0)) = 1,$$

Equation (27) was solved numerically using the software package CESDSOL, using the Newton-Raphson algorithm with compactification. The solution of the Baby Skyrmion configuration was used as an initial approximation. In equation (27) was chosen m = 1.



(a) Function f(r) with different ω



(b) Potential $\phi 3$ with different ω

Figure 6: Solutions of the equation (27)

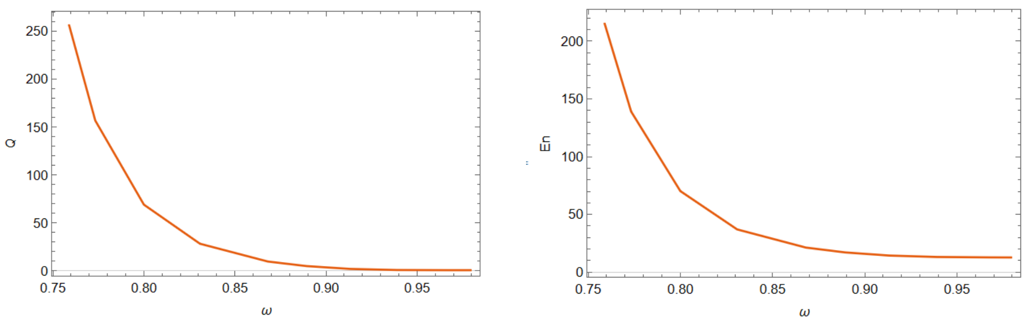


Figure 7: $Q_{SO(2)}$ and E as a functions of w

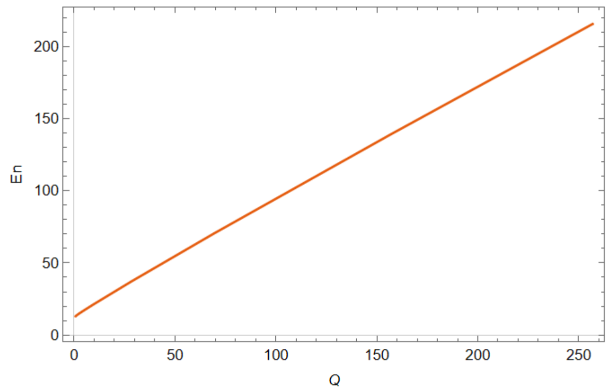


Figure 8: E as a function of $Q_{SO(2)}$

From Figure 6a we see how function f(r) changing with changing of ω. When $\omega < 0.8$ function has three regions, $f(r) \mapsto \frac{\pi}{2}$, $f(r) = \frac{\pi}{2}$ and $f(r) \mapsto 0$ (thick-wall limit). When ω increases, value of function f(r) rapidly falls to zero (thin-wall limit). As we can see from Figure 7, when $\omega \mapsto m$, then $Q_{SO(2)} \mapsto 0$, and as $\omega \mapsto \frac{1}{\sqrt{2}} - Q_{SO(2)} \mapsto \infty$. And, as expected, there is a linear dependence of energy on charge, as shown in Figure 8.

8 Conclusion

In this report were analyzed different soliton solutions, in O(3) sigma model without potential and non-linear O(3) sigma model, Q-Lumps. As was demonstrated, due to topological nontrivial structure of space on which live fileds, we can easily find soliton solution by projection of $S^2 \mapsto \mathbb{CP}^1$. It will also be interesting to consider O(4) sigma model with 6-potential and interaction of two such configurations, or spin-orbit interaction of Q-lumps in future reports.

References

[1] Shnir Y. M. Topological and Non-Topological Solitons in Scalar Field Theories. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2018.

[2] Rubakov V. A. Classical theory of gauge fields. Princeton, NJ: Princeton Univ., 2002.

[3] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, Modern Geometry. Methods and Applications. Part 1, Springer Verlag, GTM 93, 1984

[4] R. S. Ward, Topological Q-Solitons. arXiv:hep-th/0302045

[5] Paul Sutcliffe, Q-Lump scattering. arXiv:2304.05521v1

[6] M. Haberichter, Classically spinning and isospinning non-linear σ-model solitons, PhD diss., University of Manchester, 2013