JOINT INSTITUTE FOR NUCLEAR RESEARCH
Bogolyubov Laboratory of Theoretical Physics

FINAL REPORT
NUMERICAL METHODS IN THEORY OF TOPOLOGICAL SOLITONS

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O(3) sigma-model

We begin with the well-known field model which has a soliton solution - sigma-model. Here we have multiplet of scalar fields - mappings:

$$\phi^a \, : \, M \, \rightarrow \, \Phi, \tag{1}$$

where M - a manifold, e.g. spacetime, Φ - target space - arbitrary Riemann manifold, which mostly is taken to be Lie group or symmetric space. In our case $M = \mathbb{R}^{1,2}$ and $\Phi = O(3)$.

The general structure of the Lagrangian of the sigma-model in d + 1 dimension has a form of

$$L = \frac{1}{2} \int_M \partial_\mu \phi^a \partial^\mu \phi^b g_{ab}(\phi) d^d x, \tag{2}$$

where $g_{ab}(\phi)$ is a metric tensor on the target space which is dependent on the field (see [1]). In the case of the flat target space we obviously has identity as a metric tensor.

As I mentioned we will work with O(3) group, which is isomorphic to S^2, what introduces the condition $\phi_a \phi^a = 1$. Furthermore, we will restrict our consideration to static situation and forget about time dependency. If we recall stereographic projection $\mathbb{C} \cong S^2$ we get to a map $\phi^a \; : \; S^2 \; \cong \; \mathbb{C} \; \cong \; \mathbb{R}^2 \cup \{\infty\} \; \rightarrow \; O(3) \; \cong \; S^2$. It is well-known that such maps are described topologically by the second homotopy group $\pi_2(S^2) \cong \mathbb{Z}$. So we have the infinite number of non - equivalent homotopy classes, which can be figured out intuitively as maps from sphere to sphere with different winding numbers.

We want to watch how such topological features affects the system.

To start off, I found the transformation to a more convenient coordinates on the target space mapping it into complex projective space $\mathbb{CP}^1$ with conjugate variables

$$\left(W = \frac{\phi^1 + i\phi^2}{1 - \phi^3}, \quad W^* \right), \quad \text{for the north pole projection,} \tag{3}$$

$$\left(W = \frac{\phi^1 + i\phi^2}{1 + \phi^3}, \quad W^* \right), \quad \text{for the south pole projection.} \tag{4}$$

And inverse transformations

$$(\phi^1, \phi^2, \phi^3) = \left(\frac{W + W^*}{1 + |W|^2}, i\frac{W^* - W}{1 + |W|^2}, \frac{|W|^2 - 1}{1 + |W|^2} \right), \quad \text{-North pole,} \tag{5}$$

$$(\phi^1, \phi^2, \phi^3) = \left(\frac{W + W^*}{1 + |W|^2}, i\frac{W^* - W}{1 + |W|^2}, \frac{1 - |W|^2}{1 + |W|^2} \right), \quad \text{-South pole.} \tag{6}$$

One can show [1] that the N-soliton solution can be constructed by the holomorphic map of a form

$$W(z) = \frac{P(z)}{Q(z)}, \tag{7}$$

where z is a complex coordinate on the spatial part of a world-sheet, $P(z)$, $Q(z)$ are the polynomials of the degree not greater that N (degree of one's must be exactly N).

For instance, we obtained 2-soliton solution using a transformation

$$W(z) = \frac{(z-a)(z-c)}{(z-b)(z-d)}, \tag{8}$$

with a,b,c,d be constants. The profile of energy density is shown in Fig. 1

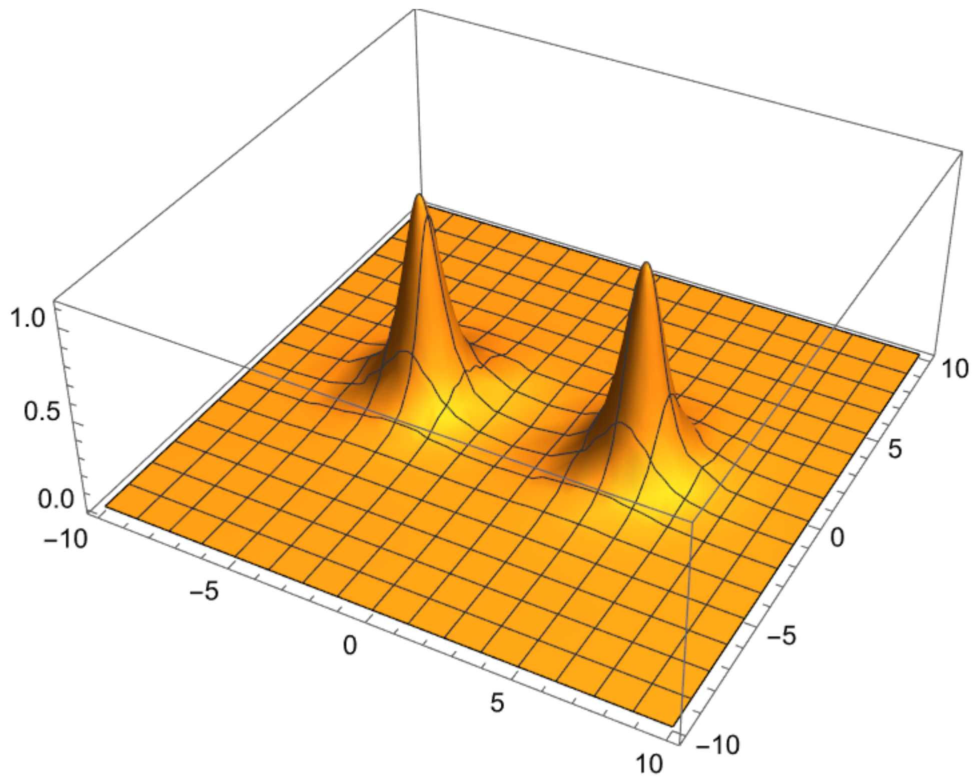

Figure 1: Energy distribution of the 2-soliton solution to O(3) sigma-model

Then I found out that more convenient form of the transformation $W(z)$ to get the manually prepared system of solitons is

$$W(z) = A\left[\sum_{k=1}^{N} \frac{1}{z-z_k}\right]^{-1}, \tag{9}$$

Here are two different 8-soliton solutions

$$W_8^{(1)} = \frac{1}{\frac{1}{z-35}+\frac{1}{z-25}+\frac{1}{z-15}+\frac{1}{z-5}+\frac{1}{z+5}+\frac{1}{z+15}+\frac{1}{z+25}+\frac{1}{z+35}}, \tag{10}$$

$$W_8^{(2)} = \frac{4}{\frac{1}{z}+\frac{1}{z+1/2-i}+\frac{1}{z-1/2-i}+\frac{1}{z-1}+\frac{1}{z+1}+\frac{1}{z+3/2+i}+\frac{1}{z-3/2+i}+\frac{1}{z-2i}}. \tag{11}$$

Both are presented in pictures below.

It worth notice that the map from the south pole can be obtained by taking the conjugation and applying the inversion transformation:

$$W_S = \frac{1}{W_N^*} \tag{12}$$

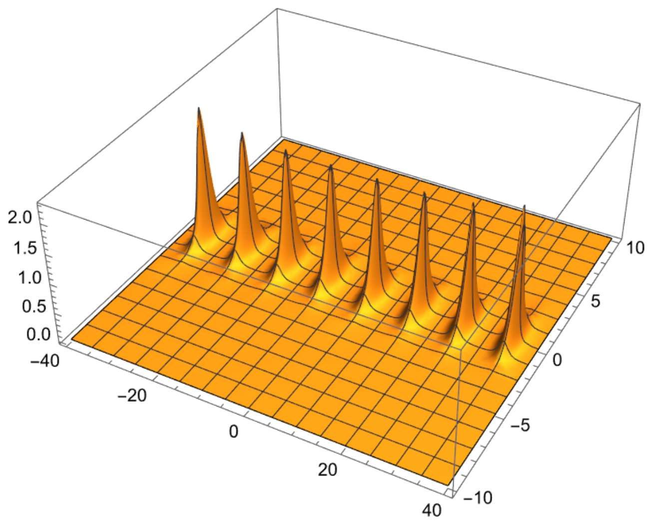

Figure 2: Energy distribution for 8-soliton solution characterised by the map $W_8^{(1)}$

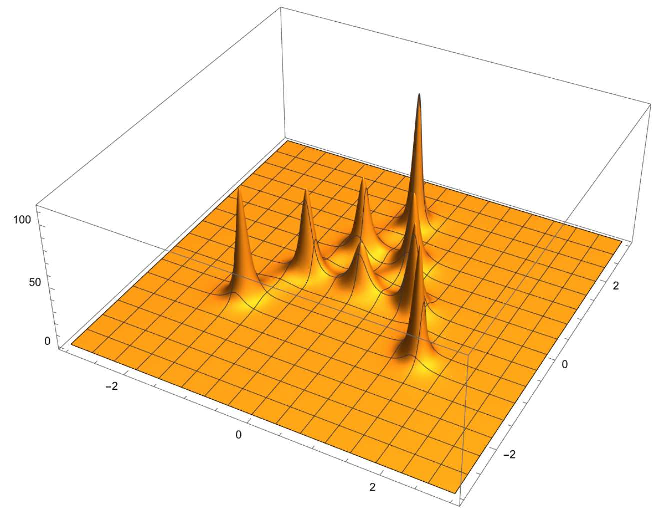

Figure 3: Energy distribution for 8-soliton solution characterised by the map $W_8^{(2)}$

Baby Skyrme model

The action of the Skyrme model is build upon the one of sigma-model with fields $\phi^a : M \to S^2$. The density of Lagrangian in the case of flat space [1] is given by

$$L = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi_a - \frac{1}{4}(\epsilon_{abc}\phi^a \partial_\mu \psi^b \partial_\nu \phi^c)^2 + V(\phi), \tag{13}$$

where $V(\phi)$ - "arbitrary" potential, which ordinary takes the form $V(\phi) = -\mu^2(1-\phi_3)$, $\mu = const$. The constrain $\phi_a \phi^a = 1$ is remained.

The map ϕ^a again may be interpreted as the winding number. So, for example, when this number is equal to 1, the field must be turned once during the parallel transfer along a large circle on S^2 passing through the north pole.

We begin with consideration of so-called Baby Skyrme model which is determined on the 2 + 1 world-sheet.

The solution to this model due to symmetry reasons can be found with anzatz

$$\psi_1 = \cos\theta \sin f(r), \quad \phi_2 = \sin\theta \sin f(r), \quad \phi_3 = \cos f(r). \tag{14}$$

Then the border conditions have a form

$$f(0) = \pi, \qquad f(\infty) = 0. \tag{15}$$

It was checked that intuitive consideration leads to the correct result of the solution with the winding number equal to 1:

$$Q = -\frac{1}{8\pi}\int \epsilon_{abc}\epsilon_{ij}\phi^a \partial_i \phi^b \partial_j \phi^c dxdy =$$
$$= \frac{1}{4\pi}\int \epsilon_{abc}\phi^a \phi^b_{,r}\phi^c_{,\theta} = \frac{1}{4\pi}\int f'(r)\sin f(r) dr d\theta = \frac{1}{2}\cos f(r)\Big|_0^\infty = 1 \tag{16}$$

The energy of the system can be found using the standard Legendre transformation

$$E = 2\pi \int_0^\infty \left[\frac{1}{2} \left(f'(r)\right)^2 + \frac{\sin^2 f(r)}{2r^2} \left(\left(f'(r)\right)^2 + 1 \right) + \mu^2 (1 - \cos f(r)) \right] r dr \tag{17}$$

The first variation of action will lead us to the equation of motion for the function $f(r)$

$$\left(r + \frac{\sin^2 f(r)}{r} \right) f''(r) + \left(1 - \frac{\sin^2 f(r)}{r^2} + \frac{\sin f(r) \cos f(r)}{r} f'(r) \right) f'(r) - \\ \frac{\sin f(r) \cos f(r)}{r} - r\mu^2 \sin f(r) = 0. \tag{18}$$

The numerical solutions of it are shown below in fig. 4.

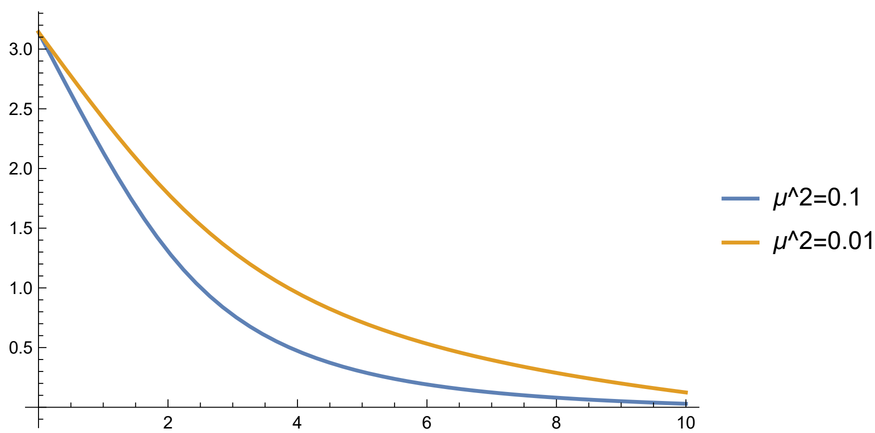

Figure 4: The solution to Baby Skyrme model (18) for different values of μ

Skyrme model

The same actions can be performed on the original Skyrme model for 3 + 1 world-sheet [1]. The Lagrangian and the spherically symmetrical anzatz have forms

$$L = -\frac{1}{2} \mathrm{Tr} U^\dagger (\partial_\mu U) U^\dagger (\partial^\mu U) + \frac{1}{16} \mathrm{Tr} \left[(\partial_\mu U) U^\dagger, (\partial_\nu U) U^\dagger \right]^2, \tag{19}$$

$$U(r) = \exp \left(i f(r) n_a \tau^a \right)) = \cos f(r) + i n_a \tau^a \sin f(r), \tag{20}$$

where τ^a - Pauli matrices.

The energy of the system results in the next equation of motion

$$E = 4\pi \int_0^\infty \left(r^2 \left(f'(r)\right)^2 + 2 \sin^2 f(r) (1 + (f'(r))^2) + \frac{\sin^4 f(r)}{r^2} + m^2 (1 - \cos f(r)) \right) dr, \tag{21}$$

$$\delta E = 4\pi \int_0^\infty \Big(r^2 2 f'(r) \partial_r \delta f + 4 \sin f(r) \cos f(r) \delta f(r) (1 + (f'(r))^2) + \\ 2 \sin^2 f(r) 2 f'(r) \partial_r \delta f + \frac{4 \sin^3 f(r) \cos f(r)}{r^2} \delta f - m^2 \sin f(r) \delta f(r) \Big) dr \tag{22}$$

$$(r^2 + 2\sin^2 f(r))f''(r) + 2rf'(r) + \sin 2f(r)\left(2f'(r) - (f'(r))^2 - 1 - \frac{\sin^2 f(r)}{r^2}\right) + \frac{1}{2}m^2 \sin f(r) = 0, \tag{23}$$

which can be solved numerically:

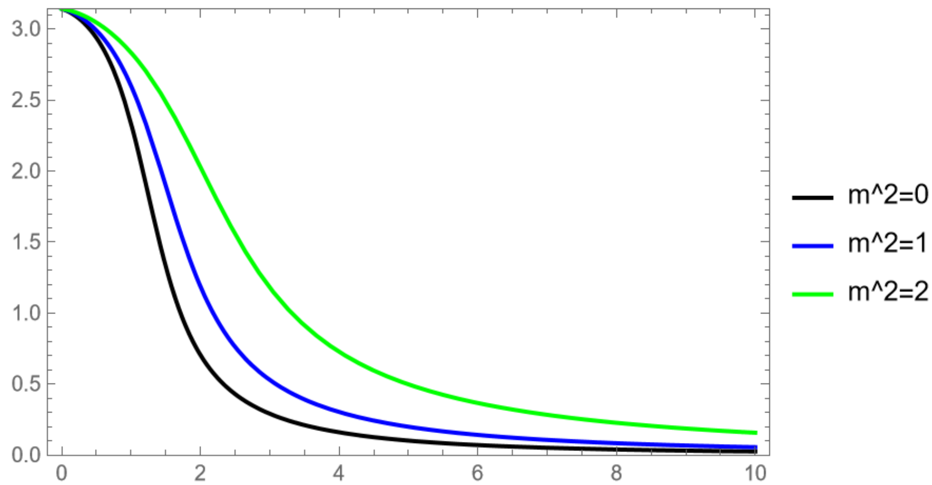

Figure 5: The solution to Skyrme model at Q = 1 for different values of parameter m.

This result corresponds to the skyrmion with the winding number Q = 1.

$$Q = -\frac{1}{24\pi^2}\int \epsilon^{ijk}\mathrm{Tr}\left(U^\dagger\partial_i U U^\dagger \partial_j U U^\dagger \partial_k U\right) d^3x, \tag{24}$$

where

$$U^\dagger \partial_i U = i\left(n_i n_a f'(r) + \frac{\sin 2f(r)}{2r}\left(\delta_{ia} - n_a n_i\right)\right)\tau^a = iF_{ia}(r)\tau^a. \tag{25}$$

Since

$$\mathrm{Tr}\left(\tau^a\tau^b\tau^c\right) = 2i\epsilon^{abc}$$

for topological charge we obtain

$$\begin{aligned} Q = -\frac{1}{24\pi^2}\int 2i^3\epsilon^{ijk}\epsilon^{abc}F_{ia}F_{jb}F_{kc}d^3x = \\ = -\frac{1}{12\pi^2}\int 6\left[f'(r)\left(\frac{\sin 2f(r)}{2r}\right)^2 + f'(r)\frac{\sin^4 f(r)}{r^2}\right]r^2 dr d\Omega = \\ = -\frac{1}{2\pi^2}\int f'(r)\frac{\sin^2 f(r)}{r^2}r^2 dr d\Omega = \\ = -\frac{4\pi}{2\pi^2}\int f'(r)\sin^2 f(r)dr = -\frac{1}{\pi}\int_0^\infty (1-\cos 2f(r))df = -\frac{1}{\pi}\left(f(\infty) - f(0)\right) = 1 \end{aligned} \tag{26}$$

Conclusion

The main result of my work was obtaining knowledge about the methods of analysing the systems with topological objects. The detailed information of spaces the theory's fields work between can significantly expand our knowledge of system and its solutions. During my study period such features were found in the models as O(3) sigma-model, euclidean Yang-Mills fields with the group SU(2) and its BPST-instanton solution, Skyrme model in two different spatial dimensions of the world-sheet.

References

[1] Y. M. Shnir, "Topological and Non-Topological Solitons in Scalar Field Theories," Cambridge University Press, 2018, ISBN 978-1-108-63625-4