JOINT INSTITUTE FOR NUCLEAR RESEARCH
Bogolyubov Laboratory of Theoretical Physics

FINAL REPORT ON THE INTEREST PROGRAMME

Numerical methods in theory of topological solitons

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1 O(3) Nonlinear Sigma Model

The *sigma model* was introduced in 1960 by Gell-Mann and Lévy [1] to describe strong interaction between pions and nucleons. In this model, a scalar field (originally noted as σ, hence the name of the model) is considered to be a map from the d-dimensional Rieamannian manifold $\mathcal{M}$ (namely a space time) onto the N-dimensional target space $\mathcal{F}$. The most general form of the Lagrangian is

$$L = \frac{1}{2} g_{ab}(\phi)\, \partial_\mu \phi^a\, \partial^\mu \phi^b, \tag{1.1}$$

where ϕ^a are real scalar fields, $a, b = 1, 2, \ldots, N$, $\mu = 1, 2, \ldots, d$, and g_{ab} is a metric tensor on target space N.

Let G be the group of symmetry that acts transitively on $\mathcal{F}$ and let H be subgroup of G that acts trivially on a certain point $\phi \in \mathcal{F}$. If there are two elements $g_1, g_2 \in G$ that have the same action on ϕ_0, then $g_1^{-1} g_2(\phi) = \phi_0$, so they belong to the same left coset of G. Since G acts transitively on $\mathcal{F}$, one can write

$$\mathcal{F} = G/H = \{gH : g \in G\}.$$

For instance, let $\vec{\phi} = (\phi_1, \ldots, \phi_N)$ to be the unit vector on the Riemann sphere S^{N-1}, so that

$$\sum_{a=1}^{N} (\phi^a \cdot \phi^a) = 1. \tag{1.2}$$

In this case, group of symmetry G is $SO(N)$, and its subgroup $H = SO(N-1)$ is the group of rotations about ϕ_0. This defines $\mathcal{F}$ as

$$\mathcal{F} = \frac{SO(N)}{SO(N-1)} = S^{N-1}. \tag{1.3}$$

This is the *O(N) nonlinear sigma model.* In this paper we will investigate $O(3)$ model which yields soliton solutions in $2+1$ dimensions. In this model Lagrangian (1.1) takes the form

$$L = \frac{1}{4}\,\partial_\mu \vec{\phi}\,\partial^\mu \vec{\phi} + \lambda \left(1 - \vec{\phi}\cdot\vec{\phi}\right), \tag{1.4}$$

where $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ is the vector of three real scalar fields ϕ_i.

1.1 Properties of stereographic projection

A sphere S^2 can be represented by a complex plane via stereographic projection. Let $N = (0,0,1)$ be a north pole of a unit sphere and $P = (\phi_1, \phi_2, \phi_3)$ is a point on this sphere. *Stereographic projection* of point P from a north pole N on the horizontal plane (ϕ_1, ϕ_2) is a point $P' = (u, w, 0)$ of intersection of the plane and a line NP (see fig. 1).

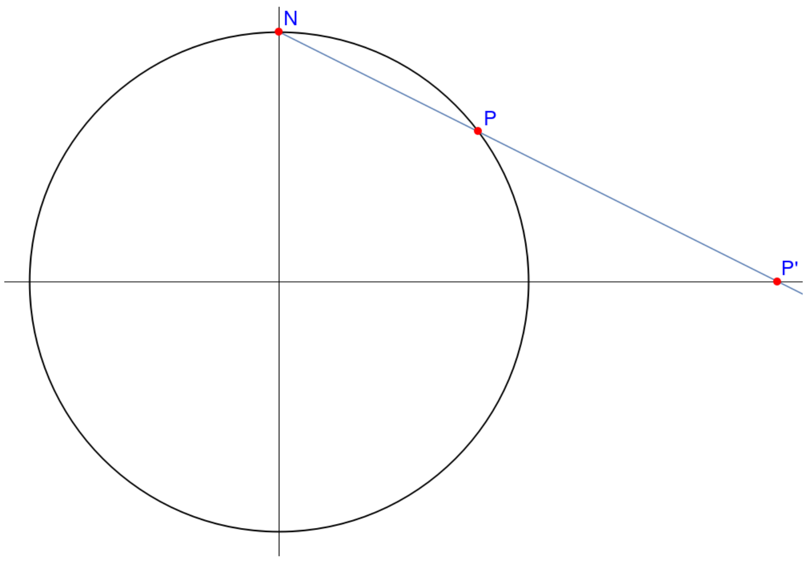


Figure 1: side view on stereographic projection from north pole

To get coordinates of P' one must consider vectors $\overrightarrow{NP} = (\phi_1, \phi_2, \phi_3 - 1)$ and $\overrightarrow{NP'} = (u, w, -1)$. It is evident that

$$\overrightarrow{NP'} = \alpha \overrightarrow{NP}, \tag{1.5}$$

where α is a real number. Comparing third components of vectors in (1.5), one obtains

$$\alpha = \frac{1}{1-\phi_3}, \tag{1.6}$$

so that

$$(u, w) = \alpha\,(\phi_1, \phi_2) = \left(\frac{\phi_1}{1-\phi_3}, \frac{\phi_2}{1-\phi_3}\right). \tag{1.7}$$

The inverse transformation can be derived from the following observation:

$$u^2 + w^2 = \frac{\phi_1^2 + \phi_2^2}{(1-\phi_3)^2} = \frac{1-\phi_3^2}{(1-\phi_3)^2} = \frac{1+\phi_3}{1-\phi_3},$$

where we used constraint (1.2). From here we get

$$\phi_3 = -\frac{1-u^2-w^2}{1+u^2+w^2}, \tag{1.8}$$

so (1.7) can be rewritten as

$$(\phi_1, \phi_2) = ((1-\phi_3)\,u, (1-\phi_3)\,w) = \left(\frac{2u}{1+u^2+w^2}, \frac{2w}{1+u^2+w^2}\right). \tag{1.9}$$

Introducing a complex variable

$$W = \frac{\phi_1 + i\phi_2}{1-\phi_3} = u + iw, \tag{1.10}$$

stereographic projection (1.8)-(1.9) can be rewritten as

$$(\phi_1, \phi_2, \phi_3) = \left(\frac{W+\overline{W}}{1+W\overline{W}}, i\frac{\overline{W}-W}{1+W\overline{W}}, -\frac{1-W\overline{W}}{1+W\overline{W}}\right), \tag{1.11}$$

where $\overline{W}$ denotes the complex conjugate of W.

Note that it is possible to make stereographic projection from the south pole $S = (0, 0, -1)$ instead of north pole N. To construct this projection, vectors $\overrightarrow{NP}$ and $\overrightarrow{NP'}$ must be replaced with vectors $\overrightarrow{SP} = (\phi_1, \phi_2, \phi_3 + 1)$ and $\overrightarrow{SP'} = (u, w, 1)$ respectively. Applying the same logic to this vectors, we can write south pole projection

$$(\phi_1, \phi_2, \phi_3) = \left(\frac{W + \overline{W}}{1 + W\overline{W}}, i\frac{\overline{W} - W}{1 + W\overline{W}}, \frac{1 - W\overline{W}}{1 + W\overline{W}} \right), \tag{1.12}$$

where

$$W = \frac{\phi_1 + i\phi_2}{1 + \phi_3} \tag{1.13}$$

The only (rather expected) difference between (1.10)-(1.11) and (1.12)-(1.13) is the sign of ϕ_3. We would use north pole projection (1.11) for definiteness.

1.2 Soliton solutions

We can now recast Lagrangian (1.4) in terms of fields $W, \overline{W}$:

$$L = \frac{\partial_\mu W \partial^\mu \overline{W}}{\left(1 + W\overline{W}\right)^2} = \frac{|\partial_t W|}{\left(1 + W\overline{W}\right)^2} - \frac{|\partial_i W|}{\left(1 + W\overline{W}\right)^2} = T - V, \tag{1.14}$$

where T and V is kinetic and potential energy densities respectively. Switching to complex variables $z = x + iy$ and $\bar{z} = x - iy$ with derivatives

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$$

allows us to write potential energy V in the following form:

$$V = \frac{|W_z|^2 + |W_{\bar{z}}|^2}{\left(1 + W\overline{W}\right)^2}, \tag{1.15}$$

where $W_z = \partial_z W$ and $W_{\bar{z}} = \partial_{\bar{z}} W$.

Variation of Lagrangian (1.14) yields the following field equation [2]

$$W_{z\bar{z}} = 2\overline{W}\frac{W_z W_{\bar{z}}}{\left(1+W\overline{W}\right)^2}. \tag{1.16}$$

Soliton solutions of this equation corresponds to the absolute minimum of the energy [3]. It can be shown [4] that the total energy E satisfies the following inequality

$$E \geqslant 4\pi\left|Q\right|, \tag{1.17}$$

where Q is topological charge, defined by the formula

$$Q = \frac{1}{4\pi}\int d^2x\; \vec{\phi}\cdot\left[\frac{\partial\vec{\phi}}{\partial x}\times\frac{\partial\vec{\phi}}{\partial y}\right] = \frac{1}{4\pi}\int\frac{\left|W_z\right|^2 - \left|W_{\bar{z}}\right|^2}{\left(1+W\overline{W}\right)^2}dzd\bar{z}. \tag{1.18}$$

For a static configuration, however, total energy can be written as

$$E = \int V dzd\bar{z} = \int\frac{\left|W_z\right|^2 + \left|W_{\bar{z}}\right|^2}{\left(1+W\overline{W}\right)^2}dzd\bar{z}, \tag{1.19}$$

so the absolute minimum of the energy corresponds to

$$W_{\bar{z}} = 0 \;\text{ for }\; Q = 4\pi E \;\text{ and }\; W_z = 0 \;\text{ for }\; Q = -4\pi E.$$

Therefore, the lower energy bound is saturated by an arbitrary holomorphic function $W(z)$ if $Q > 0$ or by arbitrary anti-holomorphic function $W(\bar{z})$ if $Q < 0$. In both cases soliton solutions of equation (1.16) are produced. The most general form of of holomorphic function $W(z)$ that delivers N-soliton solution is given by a map

$$W = \lambda\frac{P(z)}{Q(z)}, \tag{1.20}$$

where $P(z)$ and $Q(z)$ are polynomials of degree at most N and at least one of them of degree N and λ is an arbitrary complex number.

1.2.1 Two-soliton solution

As a first example, we will consider two-soliton solution given by holomorphic map

$$W(z) = \frac{(z-a)(z-c)}{(z-b)(z-d)}, \tag{1.21}$$

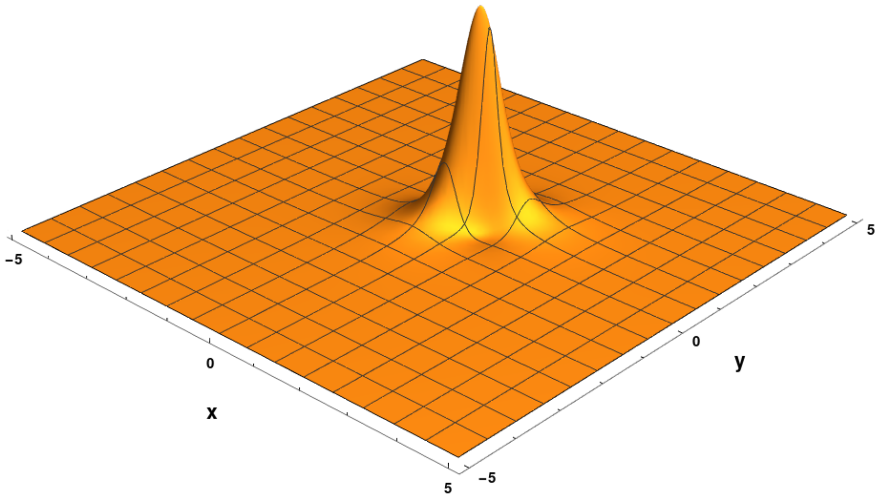
where a, b, c and d are arbitrary complex numbers. Their values define positions and characteristic scales of the solitons. Substitution of (1.21) into (1.18) yields $Q = 2$, which confirms that (1.21) delivers configuration of degree two. However, in case $a = b$, $a = d$, $c = b$ or $c = d$ map (1.21) degenerates into a map of degree one (see fig. 2a). Another specific configuration represented in fig. 2b. It corresponds to two merged solitons that forms a circular wall.

1.2.2 Eight-soliton solutions

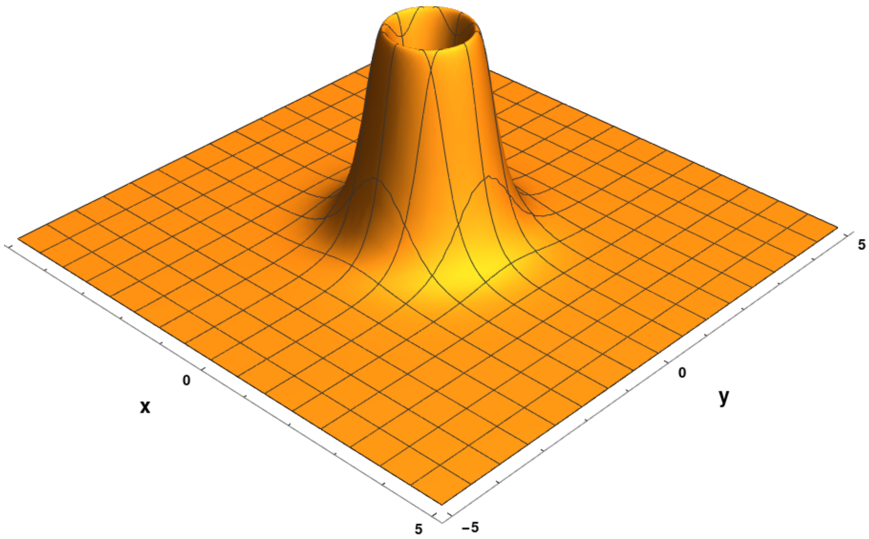
We can go further and construct soliton solutions of higher degree. For example, consider a map

$$W(z) = \frac{4}{\frac{1}{z} + \frac{1}{z+\frac{1}{2}-i} + \frac{1}{z-\frac{1}{2}-i} + \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z+\frac{3}{2}+i} + \frac{1}{z-\frac{3}{2}+i} + \frac{1}{z-2i}} \tag{1.22}$$

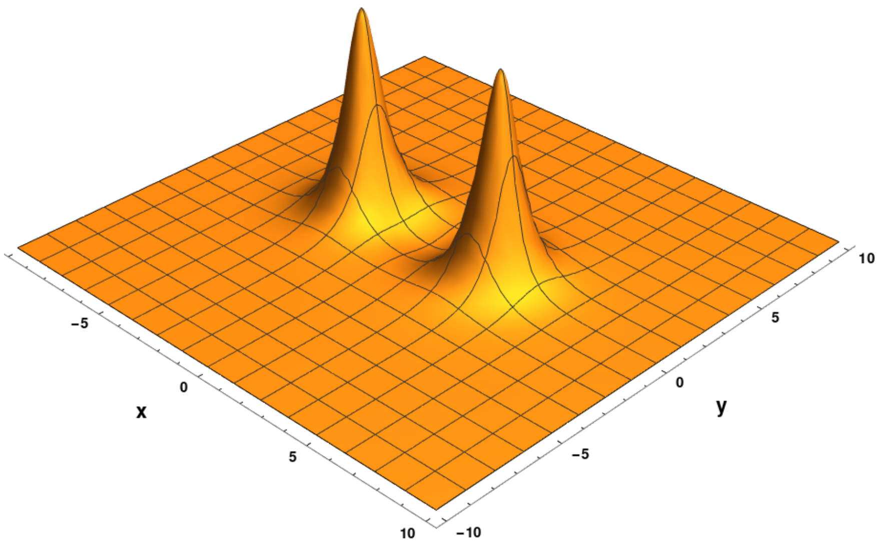
that produces eight separate solitons (see fig. 3a). Note that equivalent map for parametrization of the fields obtained from the south pole stereographic



(a) $a = c = 1,\ b = d = i$



(b) $a = -c = 1,\ b = -d = i$



(c) $a = -c = 1,\ b = -d = 4$

Figure 2: Energy density distribution of two-soliton solution (1.21) for different values of parameters a, b, c and d.

projection equals

$$W_S(\bar{z}) = \frac{1}{\overline{W}_N} = \frac{1}{4}\left(\frac{1}{\bar{z}} + \frac{1}{\bar{z}+\frac{1}{2}+i} + \frac{1}{\bar{z}-\frac{1}{2}+i} + \frac{1}{\bar{z}-1} + \frac{1}{\bar{z}+1} + \frac{1}{\bar{z}+\frac{3}{2}-i} + \frac{1}{\bar{z}-\frac{3}{2}-i} + \frac{1}{\bar{z}+2i}\right). \quad (1.23)$$

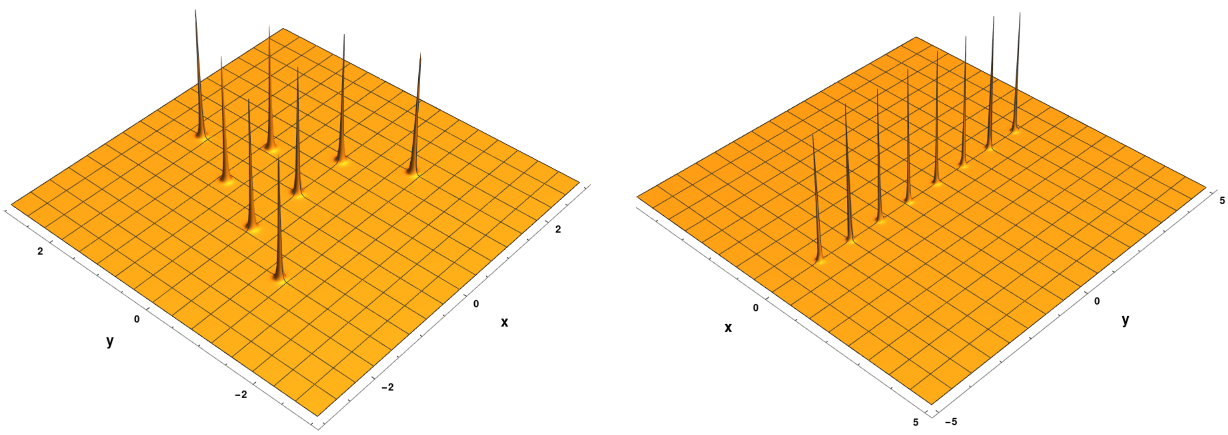
Indeed, from (1.11) follows

$$\phi_3 = -\frac{1-|W_N|^2}{1+|W_N|^2}.$$

Substituting this into (1.13), from (1.10) we obtain

$$W_S = \frac{\phi_1 + i\phi_2}{1+\phi_3} = \frac{1-\phi_3}{1+\phi_3} W_N = \frac{W_N}{|W_N|^2} = \frac{1}{\overline{W}_N}, \quad (1.24)$$

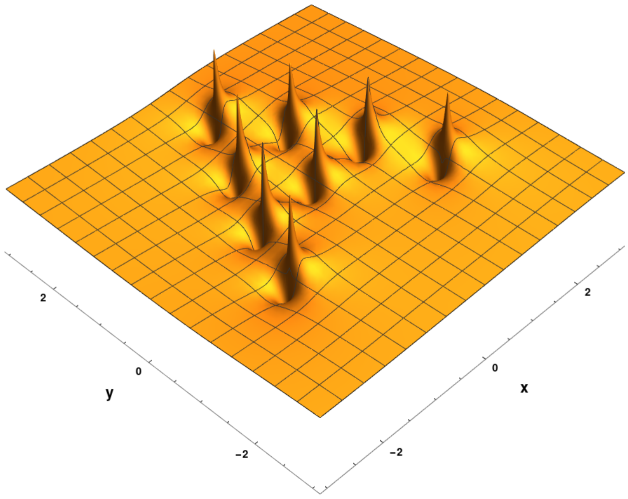
which is equivalent to (1.23).



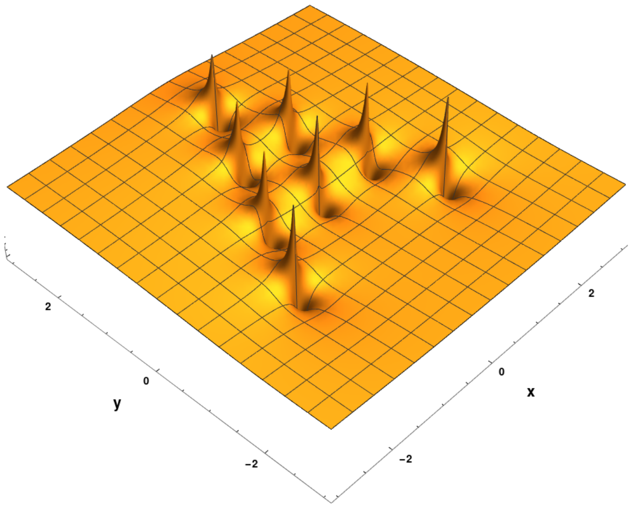
(a) Configuration for map (1.22) (b) Configuration for map (1.25)

Figure 3: Energy density distribution of eight-soliton solutions (1.22) and (1.25)

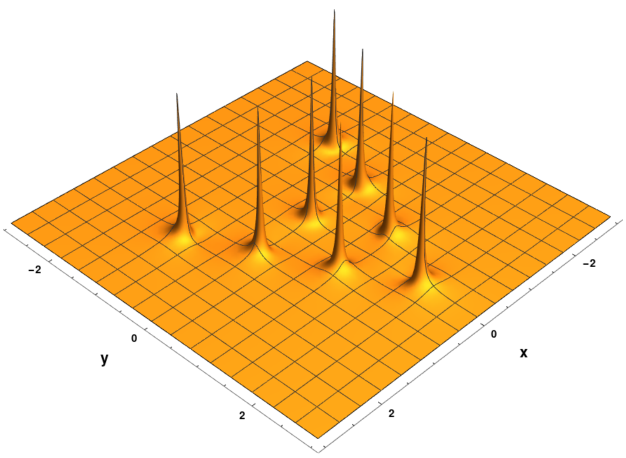
Note that solitons generated by map (1.23) are placed in the poles of the function $W_S(\bar{z})$. This implies that we can define positions of solitons by setting values of poles of the map W_S or W_N. For instance, a chain of eight aligned solitons placed along x axis would be generated by one of the



(a) Field component ϕ_1



(b) Field component ϕ_2



(c) Field component ϕ_3

Figure 4: The field components of the configuration (1.25)

following maps:

$$
\begin{aligned}
W_N =& \frac{\lambda}{\sum\limits_{n=0}^{7} \frac{1}{z-(n+d)}}, \\
W_S =& \frac{1}{\lambda} \sum_{n=0}^{7} \frac{1}{\bar{z}-(n+d)},
\end{aligned} \tag{1.25}
$$

where d is defines position of the first soliton in the chain and λ is arbitrary parameter. For the figure 3b we choosed $\lambda = 4$ and $d = -7/2$.

2 Skyrmions

One of the first field theories that supports soliton solutions was the Skyrme model [5]. Originally this model was aimed to describe baryons as topological solitons, consequently called *skyrmions*. Today Skyrme model has a lot of applications in different areas of study.

2.1 Baby Skyrme Model

We will start our review on Skyrme theory with the model of planar skyrmions in $2+1$ dimensions, known as baby Skyrme model. Lagrangian density of this model is

$$
L = \frac{1}{2}\left(\partial_\mu \vec{\phi}\right)^2 - \frac{1}{4}\left(\vec{\phi}\cdot\left[\frac{\partial\vec{\phi}}{\partial x}\times\frac{\partial\vec{\phi}}{\partial y}\right]\right)^2 - \mu^2\left(1-\phi_3\right), \tag{2.1}
$$

where μ is a rescaled mass parameter and the three dimensional vector of real scalar fields $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ take values on the unit sphere, $\vec{\phi}\cdot\vec{\phi} = 1$. That is, the field is a topological map $\phi : S^2 \to S^2$, so the model supports soliton solutions classified by topological charge (1.18):

$$Q = \frac{1}{4\pi} \int d^2x \, \vec{\phi} \cdot \left[\frac{\partial \vec{\phi}}{\partial x} \times \frac{\partial \vec{\phi}}{\partial y} \right]. \tag{2.2}$$

Soliton solutions minimize static energy functional

$$E = \int d^2x \left\{ \frac{1}{2} \left(\partial_i \vec{\phi} \right)^2 + \frac{1}{4} \left(\vec{\phi} \cdot \left[\frac{\partial \vec{\phi}}{\partial x} \times \frac{\partial \vec{\phi}}{\partial y} \right] \right)^2 + \mu^2 \left(1 - \phi_3 \right) \right\} \tag{2.3}$$

that satisfies inequality (1.17). However, solitons in the model (2.1) never saturates this bound, i.e. $E \neq 4\pi |Q|$. This forces us to use another simplification. In case of $Q = 1$ soliton isorotational $O(2)$ symmetry takes place, so the following hedgehog ansatz can be considered:

$$\vec{\phi} = (\cos\theta \sin f(r), \sin\theta \sin f(r), \cos f(r)), \tag{2.4}$$

where r and θ are polar coordinates and $f(r)$ is a monotonically decreasing function. Since the energy must be finite on the spacial infinity, the field must approach the vacuum state $\vec{\phi}_{vac} = (0, 0, 1)$ at $r \to \infty$, which means that $f(\infty) = 0$.

Substituting (2.4) in (2.3), we get a new parametrization of energy functional in terms of $f(r)$:

$$E = 2\pi \int_0^\infty r dr \left(\frac{1}{2} f'^2 + \frac{\sin^2 f}{2r^2} \left(f'^2 + 1 \right) + \mu^2 \left(1 - \cos f \right) \right). \tag{2.5}$$

Variation of this functional with respect to f yields the following equation

$$\left(r + \frac{\sin^2 f}{r} \right) f'' + \left(1 - \frac{\sin^2 f}{r^2} + \frac{f' \sin f \cos f}{r} \right) f' - \frac{\sin f \cos f}{r} - r\mu^2 \sin f = 0 \tag{2.6}$$

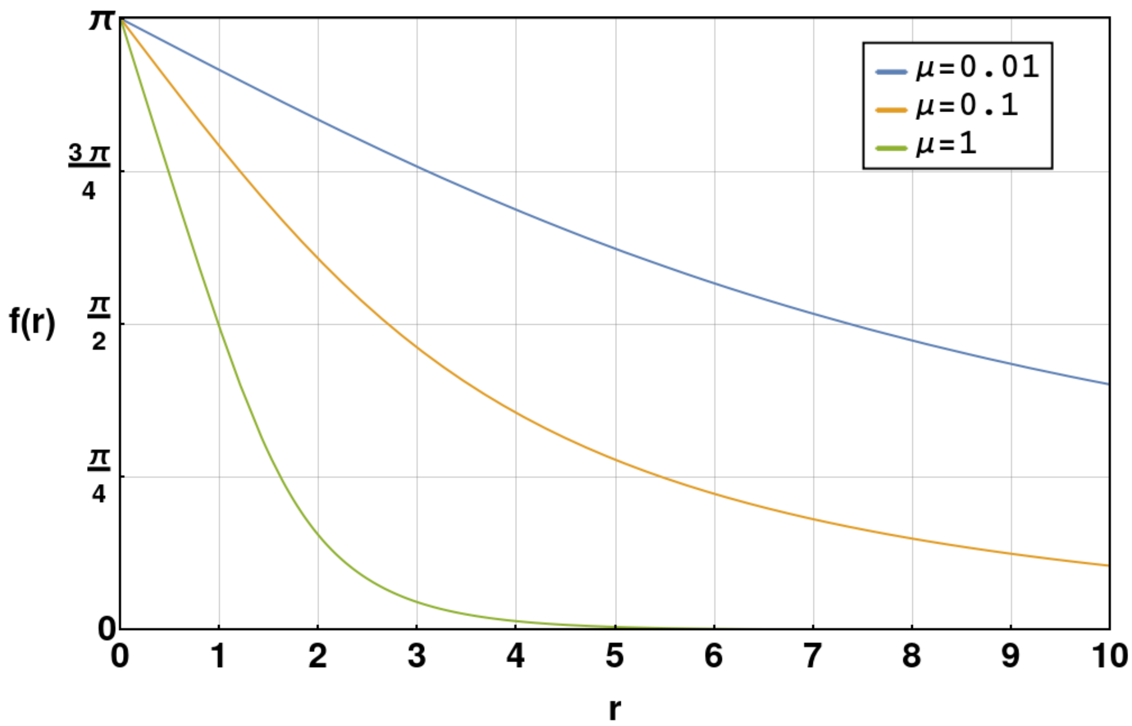


Figure 5: Plot of function $f(r)$ of the $Q = 1$ baby skyrmion with different values of parameter μ.

with $f(\infty) = 0$. To find value of f at the origin, lets check the value of topological charge (2.2):

$$\begin{aligned}
Q =& \frac{1}{4\pi}\int d^2x\,\vec{\phi}\cdot\left[\partial_x\vec{\phi}\times\partial_y\vec{\phi}\right] = \\
& \frac{1}{4\pi}\int rdrd\theta\,\vec{\phi}\cdot\left[\left(\cos\theta\partial_r\vec{\phi}-\frac{\sin\theta}{r}\partial_\theta\vec{\phi}\right)\times\left(\sin\theta\partial_r\vec{\phi}+\frac{\cos\theta}{r}\partial_\theta\vec{\phi}\right)\right] = \\
& \frac{1}{2}\int\limits_0^\infty dr\,\vec{\phi}\cdot\left[\partial_r\vec{\phi}\times\partial_\theta\vec{\phi}\right] = -\frac{1}{2}\int\limits_0^\infty dr\,f'\sin f = \frac{1}{2}\int\limits_0^\infty \frac{\partial}{\partial r}(\cos f)\,dr = \\
& \frac{1}{2}\left(1-\cos f(0)\right).
\end{aligned} \tag{2.7}$$

As we can see, $Q = 1$ if $f(0) = \pi$. Results of numerical integration of equation (2.6) with boundary conditions

$$f(\infty) = 0, \quad f(0) = \pi \tag{2.8}$$

is given on the figure 5.

2.2 Skyrme model

We can proceed further and consider Skyrme model in 3 + 1 dimensions. Lagrangian density of this model is

$$L = -\frac{1}{2}\,\mathrm{tr}\left[\left(U^\dagger\partial_\mu U\right)\left(U^\dagger\partial^\mu U\right)\right] + \frac{1}{16}\,\mathrm{tr}\left[\left(\partial_\mu U\right)U^\dagger, \left(\partial_\nu U\right)U^\dagger\right]^2 + m^2\,\mathrm{tr}\left(U - \mathbb{I}\right), \tag{2.9}$$

where $U \in SU(2)$ is the Skyrme field – a unitary unimodular matrix, and m is the mass of this field. It can be written in terms of four scalar fields (σ, π^a):

$$U = \sigma + i\pi^a \cdot \tau^a, \tag{2.10}$$

where τ^a are the three Pauli matrices.

Matrix U is restricted to the surface of the sphere S^3, i.e.

$$U \to \mathbb{I} \;\; \text{as} \;\; r \to \infty.$$

Therefore, field U is a map from coordinate space S^3 to the $SU(2)$ group space, which is isomorphic to S^3. This map is characterized by topological charge [2]

$$Q = -\frac{1}{24\pi^2}\int d^3x\varepsilon^{ijk}\,\mathrm{tr}\left[L_i L_j L_k\right], \tag{2.11}$$

where $L_i = U^\dagger\left(\partial_i U\right)$ is a left-handed $su(2)$ current. Static energy functional of the model is given by a formula

$$E = -\int d^3x\left\{\frac{1}{2}\,\mathrm{tr}\left[L_i L_i\right] + \frac{1}{16}\,\mathrm{tr}\left[L_i, L_j\right]^2 + m^2\,\mathrm{tr}\left(U - \mathbb{I}\right)\right\}. \tag{2.12}$$

Just like in baby Skyrme model, we can make use of the rotational symmetry of the $Q = 1$ configuration and consider static hedgehog ansatz

$$U(\mathbf{r}) = \cos f(r) + i \sin f(r) \hat{r}^a \cdot \tau^a, \tag{2.13}$$

where $\hat{r}^a = r^a/r$ and $f(r)$ is a real valued monotonically decreasing function. Boundary conditions on this function is defined by the value of topological charge (2.11).

To calculate this value, we should first substitute ansatz (2.13) in the definition of current L_i:

$$\begin{aligned} L_i &= U^\dagger (\partial_i U) \\ &= i\tau^a \left(\hat{r}^a \hat{r}_i f' + \frac{\delta_{ia} - \hat{r}^a \hat{r}_i}{r} \sin f \cos f + \varepsilon_{iab} \frac{\hat{r}^b}{r} \sin^2 f \right) = i\tau^a l_{ai}. \end{aligned} \tag{2.14}$$

Our second step is to simplify the integrand in (2.11). To do this, we make use of the property of Pauli matrices

$$\mathrm{tr}\left(\tau^a \tau^b \tau^c\right) = 2i\varepsilon^{abc},$$

which allows us to write

$$\varepsilon^{ijk} \, \mathrm{tr}\left(L_i L_j L_k\right) = 2i^4 \varepsilon^{ijk} \varepsilon^{abc} l_{ai} l_{bj} l_{ck} = 2\varepsilon^{ijk} \varepsilon^{abc} \left(s_{ai} + a_{ai}\right)\left(s_{bj} + a_{bj}\right)\left(s_{ck} + a_{ck}\right), \tag{2.15}$$

where

$$s_{ai} = s_{ia} = \left(f' - \frac{\sin f \cos f}{r} \right) \hat{r}_a \hat{r}_i + \frac{\sin f \cos f}{r} \delta_{ia} = R r_{ai} + d\delta_{ai}, \tag{2.16}$$

$$a_{ai} = -a_{ia} = \varepsilon_{ian} \frac{\hat{r}^n}{r} \sin^2 f = a\varepsilon_{ian} \hat{r}^n, \tag{2.17}$$

Recall that

$$
\begin{gathered}
r_{nn} = \frac{r_n r_n}{r^2} = 1, \quad r^{ai} a_{ai} = 0, \quad \varepsilon^{abc} r_{bc} = 0, \quad \varepsilon^{abc} r_{ai} r_{bj} = 0, \\
\varepsilon^{ijk} \varepsilon^{abc} a_{ck} r_{bj} \delta_{ai} = a \varepsilon^{ijk} \varepsilon^{ibc} \varepsilon_{kcn} r_{bj} \hat{r}^n = \left(\delta^i_c \delta^j_n - \delta^i_n \delta^j_c \right) \varepsilon^{ibc} r_{bj} \hat{r}^n = 0, \\
\varepsilon^{ijk} \varepsilon^{abc} a_{ai} a_{bj} = a^2 \varepsilon^{ijk} \varepsilon^{abc} \varepsilon_{ain} \varepsilon_{bjm} r^{mn} = a^2 \left(\delta^k_m \delta^c_n + \delta^k_n \delta^c_m \right) r^{mn} = 2a^2 r^{kc}, \\
\varepsilon^{ijk} \varepsilon^{abc} \delta_{ai} r_{bj} = \left(\delta^{jb} \delta^{kc} - \delta^{jc} \delta^{kb} \right) r_{bj} = \delta^{kc} - r^{kc}, \\
\varepsilon^{ijk} \varepsilon^{abc} \delta_{ai} \delta_{bj} a_{ck} = 2 \delta^{kc} a_{ck} = 0,
\end{gathered}
$$

so the following relations are satisfied

$$
\begin{aligned}
\varepsilon^{ijk} \varepsilon^{abc} s_{ai} s_{bj} a_{ck} &= 0, \\
\varepsilon^{ijk} \varepsilon^{abc} s_{ai} a_{bj} a_{ck} &= 2a^2 r^{ai} s_{ai} = 2a^2 \left(R + d \right), \\
\varepsilon^{ijk} \varepsilon^{abc} a_{ai} a_{bj} a_{ck} &= 2a^2 r^{ai} a_{ai} = 0, \\
\varepsilon^{ijk} \varepsilon^{abc} s_{ai} s_{bj} s_{ck} &= \varepsilon^{ijk} \varepsilon^{abc} \left(dR r_{ai} \delta_{bj} + dR r_{bj} \delta_{ai} + d^2 \delta_{ai} \delta_{bj} \right) s_{ck} = \\
\varepsilon^{ijk} \varepsilon^{abc} d^2 \left(\delta_{ai} \delta_{bj} (R r_{ck} + d \delta_{ck}) + \left(R r_{ai} \delta_{bj} + R r_{bj} \delta_{ai} \right) \delta_{ck} \right) &= 6 d^2 \left(R + d \right).
\end{aligned}
\tag{2.18}
$$

Using this relations, we can expand expression (2.15) to get the following result:

$$
\begin{gathered}
\varepsilon^{ijk} \operatorname{tr} \left(L_i L_j L_k \right) = \\
4d^2 \left(R + d \right) + 3 \cdot 4a^2 \left(R + d \right) = 6 \left(R + d \right) \left(a^2 + d^2 \right) = 12 f' \frac{\sin^2 f}{r^2}.
\end{gathered}
\tag{2.19}
$$

Now topological charge (2.11) can be easily calculated:

$$
\begin{aligned}
Q &= -\frac{1}{24\pi^2} \int r^2 \sin\theta \, dr d\theta d\varphi \cdot 12 f' \frac{\sin^2 f}{r^2} \\
&= -\frac{2}{\pi} \int dr \, f' \sin^2 f = -\frac{1}{\pi} \left[f(r) - \frac{\sin 2f(r)}{2} \right]_0^\infty .
\end{aligned}
\tag{2.20}
$$

It is evident that for Q to be equal unity, $f(r)$ must satisfy following boundary conditions:

$$f(0) = \pi, \quad f(\infty) = 0. \tag{2.21}$$

Total energy functional of the static hedgehog configuration can be written as

$$E = 4\pi \int_0^\infty dr \left(r^2 f'^2 + 2\sin^2 f \left(1 + f'^2\right) + \frac{\sin^4 f}{r^2} + 4r^2 m^2 \left(1 - \cos f\right) \right) \tag{2.22}$$

and corresponding variational equation

$$\left(r^2 + 2\sin^2 f\right) f'' + 2rf' - \sin 2f \left(1 - f'^2 + \frac{\sin^2 f}{r^2}\right) - 4r^2 m^2 \sin f = 0. \tag{2.23}$$

This equation, supplied with boundary conditions (2.21), can be solved numerically. Resulting function $f(r)$ is presented in figure 6.

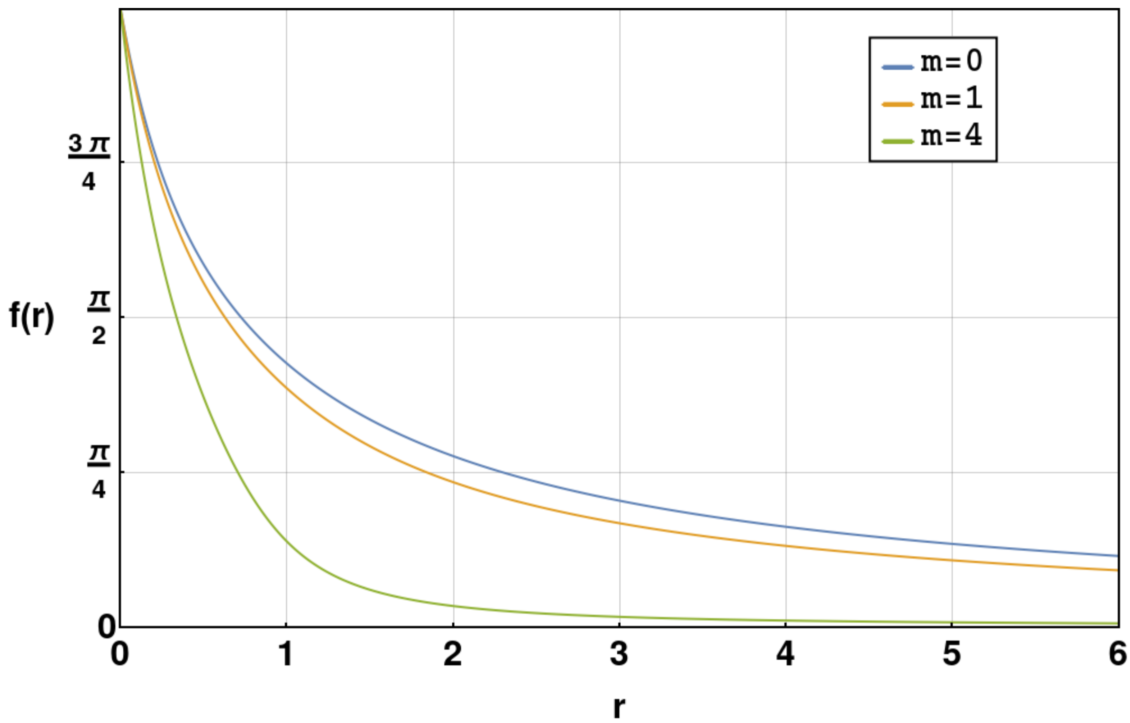


Figure 6: Plot of function $f(r)$ for skyrmions of different mass m.

A Numerical solution

In this section we will explain method used to find numerical solutions of equations (2.6) and (2.23).

Consider equation (2.6). First, we change domain of integration from $[0, \infty)$ to $[0, 1)$ by transformation

$$r = \frac{t}{1-t}, \quad t \in [0, 1). \tag{A.1}$$

which changes the form of the equation

$$\begin{aligned}
&\left(\frac{t}{1-t} + \frac{1-t}{t}\sin^2 v(t)\right)\left((1-t)^4 v''(t) - 2(1-t)^3 v'(t)\right) \\
&+ \left(1 - \frac{(1-t)^2}{t^2}\sin^2 v(t) + \frac{(1-t)^3}{t} v'(t)\sin v(t)\cos v(t)\right)(1-t)^2 v'(t) \\
&- \frac{1-t}{t}\sin v(t)\cos v(t) - \frac{1-t}{t}\mu^2 \sin v(t) = 0
\end{aligned} \tag{A.2}$$

and boundary conditions

$$v(0) = \pi, \quad v(1) = 0 \tag{A.3}$$

with $v(t) = f\left(\frac{t}{1-t}\right)$.

Next we divide interval $[0, 1)$ into $N = 512$ parts of length $h = 1/N$. This yields a set of $N+1$ points $\{t_{i=0}^N \,|\, t_{i+1} = t_i + h\}$. Values of derivatives at point t_i can be approximated by the following formulas

$$\begin{aligned}
v'(t_i) &\approx \frac{v_{i+1} - v_{i+1}}{2h}, \\
v''(t_i) &\approx \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2},
\end{aligned} \tag{A.4}$$

where $v_i \equiv v(t_i)$ and $i = 1, 2, \ldots N-1$. Substituting (A.4) into (A.2), we get discretized equations

$$\frac{(1-t_i)^3}{h^2}\left(\frac{t_i}{1-t_i}+\frac{1-t_i}{t_i}\sin^2 v_i\right)((1-t_i)(v_{i+1}-2v_i+v_{i-1})-h(v_{i+1}-v_{i-1}))$$
$$+\frac{(1-t_i)^2}{2h}(v_{i+1}-v_{i-1})\left(1-\frac{(1-t_i)^2}{t_i^2}\sin v_i+\frac{(1-t_i)^2}{2ht_i}(v_{i+1}-v_{i-1})\cos v_i\sin v_i\right)$$
$$-\frac{1-t_i}{t_i}\cos v_i\sin v_i-\mu^2\frac{t_i}{1-t_i}\sin v_i=0,\quad i=1,2,\ldots N-1, \tag{A.5}$$

where $v_0=\pi$ and $v_N=0$. This equations can be solved with respect to v_i using Newton's method with initial guess

$$v_i^{init}=\pi(1-t_i). \tag{A.6}$$

The same procedure applies to equation (2.23) for Skyrmions.

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