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FINAL REPORT ON THE INTEREST PROGRAMME

Transport of high-energy particles through the non-extensive quark-gluon plasma

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Supervisor:
Dr. Trambak Bhattacharyya

Student:
Alexandra-Gabriela Şerban
Faculty of Physics, University of Bucharest

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Contents

Abstract

A prominent area of research is represented by the study of quark-gluon plasma (QGP) formed during high-energy collisions of protons and heavy ions at BNL, at the Relativistic Heavy Ion Collider (RHIC) and at CERN, at the Large Hadron Collider (LHC). Probing the plasma medium with high-energy particles which are produced right after the collision, before the medium is actually formed, is one technique to understand the many features of QGP. The evolution into hadrons of the quarks and gluons in the plasma medium involves intrinsic fluctuations and long-range correlations. Thus, a state of thermal equilibrium is difficult to be obtained and what is attained is some kind of stationary state which is not governed by the usual Boltzmann-Gibbs statistics, but by the Tsallis statistics proposed in 1988 by Constantino Tsallis. The project aims at understanding the role of the Tsallis statistics in the study of the transport properties of high-energy particles which are traveling through QGP.

Introduction

One of the main goals of studying heavy-ion collisions at relativistic energies is to search for quark-gluon plasma (QGP) and to study its properties. The high temperature/high energy density needed to produce QGP in a laboratory environment can be achieved in proton - proton (p-p) and heavy-ion (A-A) collisions at energies which are accessible at the Large Hadron Collider (LHC) at CERN and at the Relativistic Heavy Ion Collider (RHIC) at BNL. Trying to understand the transport properties of the QGP medium through the interaction of probes with the medium brings to light useful information about the nature of the medium. There are two different evolutions that can be studied. The first one is the evolution of the QGP medium which is governed by the hydrodynamic equation [1]: $\partial_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is the energy momentum tensor. The second evolution is that of the high-energy particles which are the probes for QGP and this is the study which is of interest for us. Why are we focusing on the high-energy particles? - because they are produced very early after the collision, in the pre-equilibrium phase, and in this way they are witnesses to the entire evolution of the plasma, they do not become part of the medium when passing through it and their energy loss due to the interaction with the particles of the medium gives us information about the plasma medium. Boltzmann-Gibbs distribution functions are used to fit experimental data for the transverse momentum (p_T) distribution of particles coming from high-energy collisions experiments due to the assumption of the local thermal equilibrium. However, this model is only accurate in describing the spectra at low p_T values. Since hadronizing systems experience strong intrinsic fluctuations and long-range correlations, the usual thermal equilibrium is hard to be achieved. Instead there is a kind of stationary state. This behaviour can be treated more appropriately in the theoretical framework of the generalized non-extensive statistical mechanics proposed by Constantino Tsallis in 1988 [2]. Power-law functions [3] are used to describe very well the experimental data for the p_T spectra at RHIC [4] and LHC [5]. The mentioned applications of the distribution function emerging from Tsallis statistics are in the high energy physics domain. Nonetheless, studies in cosmology and astrophysics have also been done using these generalized statistical approaches and a few examples [6] are: entropy and formation of black holes [7], direct dark matter detection rates [8] and stellar rotational velocities [9].

The aim of our report is to understand the evolution of high-energy particles inside QGP which help us study the properties of the plasma medium. In Section 1 we start with the study of the basics of QGP [10], answering essential questions such as 'What is quark-gluon plasma?' and 'Why do we study it?'. Then, we discuss about the transport of high-energy particles inside the QGP medium. In Section 2, the evolution is presented in the usual framework of the Boltzmann-Gibbs statistics. We derive the well-known equation in the kinetic theory of gases which is the Boltzmann transport equation (BTE) [11, 12] and we solve it analytically in the relaxation time approximation (RTA) for a uniform QGP with no external force. After that, we present two experimental observables of high interest in high-energy collisions: the nuclear suppression factor, which we plot as a function of the transverse momentum p_T, and the elliptic flow. In Section 3, we discuss the evolution of the high-energy particles inside non-extensive QGP. Starting from the generalized entropy introduced by Constantino Tsallis, we derive the Tsallis probabilities which are useful in determining the phase space distribution functions of the high-energy particles. Since the non-extensive features during the collisions lead to a stationary state, not a thermal equilibrium state, we prove following Ref.[13] that there exists a modified version of the Boltzmann transport equation inspired by the Tsallis statistics that has a Tsallis-like (power-law) stationary solution. In a similar manner in which we studied the RTA for the BTE, we want to use this approximation to arrive at a solution of the modified BTE based on a perturbative approach proposed in Ref. [14]. We end our discussion with a short summary and the main conclusions.

Very common in theoretical high-energy particle physics and nuclear physics is the natural system of units which we will employ throughout this report. We set the reduced Planck constant, the speed of light and the Boltzmann constant to be 1: $\hbar = c = k_B = 1$.

Project Goals

The main objectives that we set out to achieve during the project are:

- Studying the basics of the physics of QGP;
- Analyzing the evolution of high-energy particles inside QGP;
- Connection to experimental observables.

1 Basics of QGP

The first question that arises in our minds is: what is quark-gluon plasma?

The elementary particles which describe everything around us are: fermions, which are matter particles, and bosons, which are mediator particles. The forces through which these fundamental particles interact are: the gravitational interaction, the electromagnetic interaction, the weak interaction and the strong interaction. A hadron is made of two or more quarks bound together by the strong interaction. Based on the number of constituents, hadrons are classified into: mesons, made of an even number of quarks (usually one quark and one antiquark) and baryons, made of an odd number of quarks (in general three quarks).

At high density or temperature, individual hadrons lose their identity and matter is described best in terms of its constituents: quarks and gluons. A particular quark in a hadron knows which are its partner quarks at low density. However, at high density, when the hadrons start to interpenetrate each other, a particular quark will not be able to identify its partners from lower density nuclear matter. At high temperature, similar phenomena happen: as the temperature is increased, more and more hadrons are created. The system will become dense enough and hadrons will start to interpenetrate. **The system where hadrons interpenetrate is considered quark matter, namely quark-gluon plasma (QGP), rather than matter made of individual hadrons.**

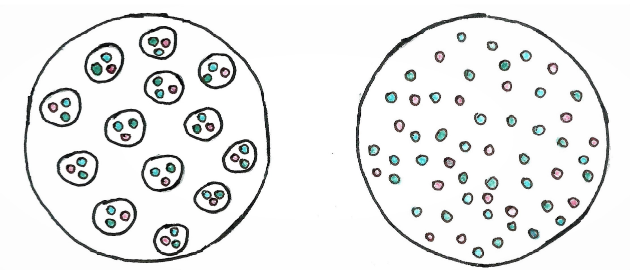


Figure 1: On the left hand side there is a representation of the nuclear matter at normal density/low temperature, while the right hand side shows the nuclear matter at high density/high temperature.

At low density or low temperature, quarks are confined within hadrons, while at high density or high temperature, quarks are deconfined. **Thus, QGP represents the deconfined state of strongly interacting matter.**

Now, having defined what quark-gluon plasma actually is, another interesting question that we can ask ourselves is: why to study quark-gluon plasma? The reason why we study QGP is to understand the microseconds old universe, the core of neutron stars and so on. High-temperature QGP was created microseconds after the Big Bang, and it is the only early universe phase transition that can be accessed in the laboratory so far. Neutron star cores also contain QGP at high density.

2 Transport of High-Energy Particles inside QGP

2.1 Boltzmann Transport Equation

High-energy particles act as effective probes to look into the properties of QGP and they are high-energy particles which are created before the QGP medium, in the pre-equilibrium phase. The evolution of their phase space distribution is described by a nonlinear integro-differential equation which is the Boltzman transport equation (BTE) which we will explicitly derive.

BTE is the basic equation in the kinetic theory of gases, satisfied by the distribution function $f(t, \mathbf{r}, \mathbf{p})$. Each monoatomic gas molecule would be a closed subsystem if collisions between molecules were negligible, and the distribution function of the molecules would satisfy Liouville's theorem, according to which,

$$\frac{df}{dt} = 0 \tag{1}$$

If the gas is in an external field $U(\mathbf{r})$ then,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{F} \cdot \frac{\partial f}{\partial \mathbf{p}} \tag{2}$$

where, $\mathbf{F} = -\nabla U(\mathbf{r})$ is the force with which the external field acts on the molecule.

In the absence of an external field, the above equation reduces to,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f \tag{3}$$

The statement of Liouville's theorem 1 is no longer valid when we take into account the collisions between the molecules of the gas. Instead, what we have is,

$$\frac{df}{dt} = C[f] \tag{4}$$

where, $C[f]$ is called the collision integral and it represents the rate of change of the distribution function due to collisions.

Making use of expression 3, we can rewrite equation 4 in the following way,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = C[f] \tag{5}$$

Our goal is to determine the form of the collision integral in order to proceed towards solving the integro-differential equation.

When two molecules collide, the values of their momenta are changed. A collision which transfers the momentum p of a molecule outside a particular range d^3p is called a loss: $\mathbf{p}, \mathbf{p}_1 \to \mathbf{p}', \mathbf{p}_1'$. In a similar manner, we can define a gain as the collision in which the momentum of a molecule, having the value outside the range d^3p, is brought to this particular range: $\mathbf{p}', \mathbf{p}_1' \to \mathbf{p}, \mathbf{p}_1$.

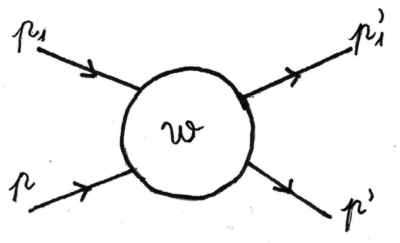


Figure 2: Elastic scattering of particles having momenta $\mathbf{p}, \mathbf{p}_1$ and $\mathbf{p}', \mathbf{p}_1'$ with transition rate w.

We can subtract the losses from the gains occurring in a volume dV per unit time. What we find is that as a result of the collisions the number of molecules is increased, per unit time, by,

$$dVd^3p \int w\left(\mathbf{p}, \mathbf{p}_1; \mathbf{p}', \mathbf{p}_1'\right) f' f_1' d^3p_1 d^3p' d^3p_1' - dVd^3p \int w\left(\mathbf{p}', \mathbf{p}_1'; \mathbf{p}, \mathbf{p}_1\right) f f_1 d^3p_1 d^3p' d^3p_1' = \\ = dVd^3p \int \left(w' f' f_1' - w f f_1\right) d^3p_1 d^3p' d^3p_1' \tag{6}$$

where we used the simplified notation,

$$\begin{cases} w = w\big(\mathbf{p}', \mathbf{p}_1'; \mathbf{p}, \mathbf{p}_1\big) \\ w' = w\big(\mathbf{p}, \mathbf{p}_1; \mathbf{p}', \mathbf{p}'\big) \end{cases} \tag{7}$$

Thus, the collision integral takes the following form,

$$C[f] = \int \big(w' f' f_1' - w f f_1\big) d^3p_1 d^3p' d^3p_1' \tag{8}$$

By using the fact that each collision is a reversible process (principle of detailed balancing),

$$\int w\big(\mathbf{p}', \mathbf{p}_1'; \mathbf{p}, \mathbf{p}_1\big) d^3p_1 d^3p' d^3p_1' = \int w\big(\mathbf{p}, \mathbf{p}_1; \mathbf{p}', \mathbf{p}_1'\big) d^3p_1 d^3p' d^3p_1' \tag{9}$$

the collision integral becomes,

$$C[f] = \int w' \left(f' f_1' - f f_1\right) d^3p_1 d^3p' d^3p_1' \tag{10}$$

Having established the form of the collision integral, we can write now the transport equation as,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = \int w' \left(f' f_1' - f f_1\right) d^3p_1 d^3p' d^3p_1' \tag{11}$$

For a qualitative study of the transport phenomena we can introduce the mean free path l, defined as the average distance traveled by a molecule between two successive collisions.

We denote by $\bar{r}$ the mean distance between molecules and assume that the mean free path is much larger than the mean distance between molecules: $l \gg \bar{r}$ in a gas.

The ratio,

$$\tau \sim \frac{l}{\bar{v}} \tag{12}$$

is called the **mean free time**.

With the mean free time introduced, we can estimate the collision integral as,

$$\boxed{C[f] \simeq -\frac{f - f_{\text{eq}}}{\tau}} \tag{13}$$

The above equation tells us that when the distribution function f is the equilibrium distribution f_{eq}, the collision integral is zero. The minus sign expresses the fact that collisions are the mechanism for reaching equilibrium and τ characterizes precisely the time scale for the system to relax to equilibrium.

2.2 Relaxation Time Approximation

The relaxation time approximation (RTA) presented above in the general framework of the kinetic theory of gases can be used in the case of uniform QGP, in absence of any external force.

$$\frac{\partial f(\mathbf{p}, t)}{\partial t} = -\frac{f - f_{\text{eq}}}{\tau} \tag{14}$$

f_{eq} is the equilibrium distribution function and τ is the relaxation time that determines the rate at which the fluctuations in the system drive it to a state of equilibrium. In this form, the equation is very easy to solve.

$$\left.\begin{aligned} \frac{\partial f}{\partial t} &= -\frac{f - f_{\text{eq}}}{\tau} \\ \frac{\partial f_{\text{eq}}}{\partial t} &= 0 \end{aligned}\right\} \implies \frac{\partial}{\partial t}(f - f_{\text{eq}}) = -\frac{f - f_{\text{eq}}}{\tau}$$

We denote $f - f_{\text{eq}} = y$.

$$\frac{\partial y}{\partial t} = -\frac{y}{\tau} \implies -\frac{1}{y}dy = -\frac{1}{\tau}dt$$

By integrating,

$$\int_{y_0}^{y} \frac{1}{y} dy = -\frac{1}{\tau} \int_0^t dt \implies \ln y - \ln y_0 = -\frac{t}{\tau} \implies \ln\left(\frac{y}{y_0}\right) = -\frac{t}{\tau} \implies \frac{y}{y_0} = e^{-\frac{t}{\tau}}$$

$$\implies y = y_0 \, e^{-\frac{t}{\tau}} \tag{15}$$

But, $y = f - f_{\text{eq}}$. Thus,

$$f - f_{\text{eq}} = \left[f(0) - f_{\text{eq}}\right] e^{-\frac{t}{\tau}} \quad ; \quad f(0) = f_{in} \tag{16}$$

$$\implies \boxed{f(\mathbf{p}, t) = f_{\text{eq}} + \left(f_{\text{in}} - f_{\text{eq}}\right) e^{-\frac{t}{\tau}}} \tag{17}$$

2.3 Connection to Experimental Observables

Transverse momentum (p_T) spectra of hadrons carry essential information about the particle production mechanism in high-energy proton - proton (p-p) and nucleus - nucleus (A-A) collisions and it is one of the main observables measured in high-energy collision events, allowing us to determine many other experimental quantities such as nuclear modification factors [15, 16] and flows [17].

1. The **nuclear suppression factor** R_{AA} is experimentally well established and it is a measure of the modification of particle production. R_{AA} expresses the suppression of high transverse momentum (p_T) hadron production in ultra-relativistic heavy-ion (A-A) collisions as compared to the scaled production from proton-proton (p-p) collisions.

$$R_{AA} = \frac{\left(d^2N/dp_T\,dy\right)^{A+A}}{N_{coll} \times \left(d^2N/dp_T\,dy\right)^{p+p}} \tag{18}$$

- $\left(d^2N/dp_T\,dy\right)^{A+A}$ = yield in nucleus - nucleus collisions;
- $\left(d^2N/dp_T\,dy\right)^{p+p}$ = yield in proton - proton collisions;
- N_{coll} = number of binary nucleon - nucleon collisions averaged over the impact parameter range of the corresponding centrality bin calculated by Glauber Monte Carlo simulation [18];
- y = rapidity.

If $R_{AA} = 1$, A-A collisions are a superposition of scaled p-p collisions. Otherwise, if $R_{AA} \neq 1$, a modification of the medium is indicated.

Another way in which we can define the nuclear suppression factor is by taking the ratio between the final and initial distributions,

$$R_{AA} = \frac{f_{\text{fin}}}{f_{\text{in}}} \tag{19}$$

where, f_{in} is the distribution of the high-energy particles immediately after their formation, while f_{fin} is the distribution of the particles after the interaction with the QGP medium.

We describe the initial distribution f_{in} of the high-energy particles using the Tsallis power-law distribution parameterized by the Tsallis non-extensivity parameter q and the Tsallis temperature T,

$$f_{\text{in}} = \left[1 + (q-1)\frac{\sqrt{p_T^2 + m^2}}{T}\right]^{-\frac{q}{q-1}} \quad , \quad m = \text{mass of the high-energy particles} \tag{20}$$

Into the Boltzmann transport equation (BTE) we plug in our chosen initial distribution and solve the equation in the relaxation time approximation (RTA) of the collision term to find out the form of final distribution f_{fin},

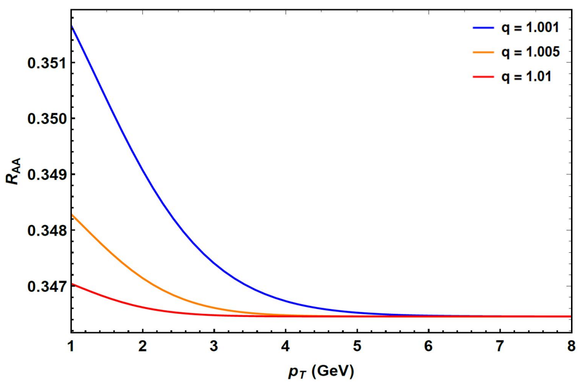
$$f_{\text{fin}} = f_{\text{eq}} + \left(f_{\text{in}} - f_{\text{eq}}\right) e^{-\frac{t}{\tau}} \quad , \quad t = \text{freeze-out time} \tag{21}$$

The equilibrium distribution f_{eq} is chosen to be the Boltzmann-Gibbs distribution,

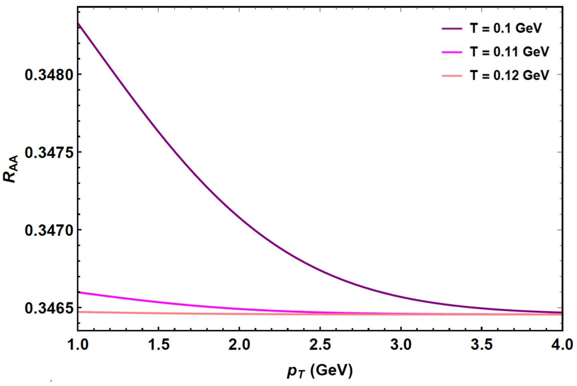
$$f_{\text{eq}} = e^{-\frac{\sqrt{p_T^2+m^2}}{T_{eq}}} \tag{22}$$

Having now established the precise forms of the initial and the final distributions, we can go back to the definition of the nuclear suppresion factor 19, which becomes [15],

$$\begin{aligned} R_{AA} &= \frac{f_{\text{eq}}}{f_{\text{in}}} + \left(1 - \frac{f_{\text{eq}}}{f_{\text{in}}}\right) e^{-\frac{t}{\tau}} \\ &= \frac{e^{-\frac{\sqrt{p_T^2+m^2}}{T_{eq}}}}{\left[1 + (q-1)\frac{\sqrt{p_T^2+m^2}}{T}\right]^{-\frac{q}{q-1}}} + \left[1 - \frac{e^{-\frac{\sqrt{p_T^2+m^2}}{T_{eq}}}}{\left[1 + (q-1)\frac{\sqrt{p_T^2+m^2}}{T}\right]^{-\frac{q}{q-1}}}\right] e^{-\frac{t}{\tau}} \end{aligned} \tag{23}$$



(a) $T = 0.13$ GeV and $T_{eq} = 0.11$ GeV



(b) $q = 1.005$ and $T_{eq} = 0.09$ GeV

Figure 3: Nuclear suppression factor R_{AA} as a function of transverse momentum p_T on the left (3.a) for different q values and on the right (3.b) for different T values. The mass is $m = 3.096$ GeV and the ratio $t/\tau = 1.06$.

2. The **elliptic flow** is a powerful probe of the initial state of the high-energy heavy ion collisons since it is sensitive to the early evolution of the system. This quantity can be theoretically obtained from the solution of the BTE.

In non-central collisions, the overlapped region of two colliding nuclei has an almond shape generating an anisotropy in the coordinate space. This anisotropy is transferred to the momentum-space of the produced particles.

Thus, the elliptic flow describes the azimuthal momentum space anisotropy of particle emission from non-central heavy ion collisions in the plane transverse to the beam direction and it is defined as the second Fourier coefficient of the azimuthal asymmetry.

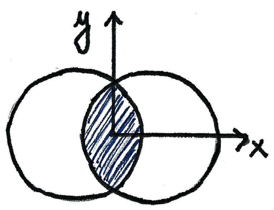


Figure 4: Almond shape overlapping region of two colliding nuclei generating an anisotropy in the coordinate space.

3 Transport of High-Energy Particles inside Non-Extensive QGP

Tsallis-like power-law functions give very good description of the p_T hadron spectra since the experimental data for the transverse momentum distribution of particles coming from proton - proton and heavy-ion collisions at the LHC and RHIC energies are described by power-law functions. This behavior is due to the fact that the hadronizing system is a few-particle system and it contains fluctuations and long-range correlations. Tsallis-like distributions can be obtained once we have the Tsallis probabilities [19].

3.1 Tsallis probabilities

In order to derive the Tsallis probabilities we start from the definition of the generalized entropy [2] with the probabilities p_i of the microstates of the system normalized to unity.

$$S = \sum_i \frac{p_i^q - p_i}{1-q} \quad , \quad \sum_i p_i = 1 \quad , \quad q \in \mathbb{R} \quad , \quad 0 < q < \infty \tag{24}$$

The parameter q is called the non-extensivity parameter. In the limit $q \to 1$ we recover the well-known Boltzmann-Gibbs entropy. The non-extensivity parameter is not just a fitting parameter, but its physical significance can be attributed to fluctuations in the temperature [20, 21].

Further, we will use the standard form of the expectation value,

$$\langle A \rangle = \sum_i p_i \, A_i \tag{25}$$

We introduce the thermodynamic potential of the grand canonical ensemble [19] which represents the Legendre transform of the fundamental thermodynamic potential $\langle H \rangle$.

$$\Omega = \langle H \rangle - T\,S - \mu \langle N \rangle \tag{26}$$

The mean energy of the system and the mean number of particles are given by,

$$\begin{cases} \langle H \rangle = \sum_i p_i \, E_i \\ \langle N \rangle = \sum_i p_i \, N_i \end{cases} \tag{27}$$

where, E_i and N_i are the energy and the number of particles in the i^{th} microscopic state of the system.

Replacing the definitions from equations 24 and 27 into equation 26 we get,

$$\Omega = \sum_i p_i \left(E_i - \mu \, N_i - T \frac{p_i^{q-1} - 1}{1-q} \right) \tag{28}$$

The unknown probabilities p_i are constrained by an additional function,

$$\varphi = \sum_i p_i - 1 = 0 \tag{29}$$

These probabilities are obtained from the second law of thermodynamics (the principle of maximum entropy). In the grand canonical ensemble the set of equilibrium probabilities p_i can be found from the constrained local extrema of the thermodynamic potential given in 26 by the method of the Lagrange multipliers.

$$\Phi = \Omega - \lambda\varphi \quad , \quad \lambda = \text{arbitrary real constant} \tag{30}$$

$$\frac{\partial \Phi}{\partial p_j} = 0 \tag{31}$$

By replacing the explicit form of the thermodynamic potential 32 and the constraint from equation 29 into relation 30 we get,

$$\Phi = \sum_i p_i \left(E_i - \mu N_i - T\frac{p_i^{q-1} - 1}{1-q} - \lambda \right) + \lambda \tag{32}$$

And now we impose the condition for obtaining the local extrema 31,

$$\begin{aligned}
\sum_i \delta_{ij} \left(E_i - \mu N_i - T\frac{p_i^{q-1} - 1}{1-q} - \lambda \right) + \sum_i p_i \left(-\frac{T}{1-q}(q-1)\, p_i^{q-2}\delta_{ij} \right) &= 0 \\
E_j - \mu N_j - \frac{T}{1-q}p_j^{q-1} + \frac{T}{1-q} - \lambda - (q-1)\frac{T}{1-q}p_j^{q-1} &= 0 \\
\frac{T}{1-q}(1+q-1)\, p_j^{q-1} &= E_j - \mu N_j + \frac{T}{1-q} - \lambda \\
q\frac{T}{1-q}p_j^{q-1} &= E_j - \mu N_j + \frac{T}{1-q} - \lambda
\end{aligned} \tag{33}$$

$$\begin{aligned}
\Longrightarrow \; p_j^{q-1} &= \frac{1-q}{q}\frac{1}{T}\left(E_j - \mu N_j + \frac{T}{1-q} - \lambda \right) \\
&= 1 + \frac{q-1}{q}\frac{\Lambda - T - E_j + \mu N_j}{T}
\end{aligned} \tag{34}$$

$$\Longrightarrow \; \boxed{p_j = \left[1 + \frac{q-1}{q}\frac{\Lambda - T - E_j + \mu N_j}{T} \right]^{\frac{1}{q-1}}} \tag{35}$$

where, $\Lambda \equiv \lambda - T$ and $\partial E_i/\partial p_i = \partial N_i/\partial p_i = 0$.

3.2 Stationary solution of the modified BTE

In hadronizing systems there are strong intrinsic fluctuations as well as long-range correlations which can lead to non-equilibrium effects. Therefore, the usual thermal equilibrium is hard to be attained and instead there is obtained a kind of power-law stationary state that can be parameterized with the help of the Tsallis-like distributions. Since the stationary solution of the conventional BTE is exponential, we need a modified BTE that yields a power-law distribution as the stationary solution. In this section we discuss a modified BTE inspired by the Tsallis statistics and its stationary solution.

The starting point is the non-extensive version of the Boltzmann equation [22], the metric used being $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$,

$$p^\mu \partial_\mu f_q^q(x, p) = C_q(x, p) \tag{36}$$

$$\begin{aligned} C_q(x, p) = \frac{1}{2} \int \frac{d^3p_1}{p_1^0} \frac{d^3p'}{p'^0} \frac{d^3p'_1}{p_1'^0} \Big\{ & h_q\left[f'_q, f'_{q1}\right] W\left(p', p'_1 \mid p, p_1\right) - \\ & - h_q\left[f_q, f_{q1}\right] W\left(p, p_1 \mid p', p'_1\right) \Big\} \end{aligned} \tag{37}$$

- $f_q(x, p) = q$ version of the corresponding phase-space distribution function;

- $C_q(x, p) = q$ collision term;

- $W\left(p, p_1 \mid p', p'_1\right)$ = transition rate between the two-particle state with initial four-momenta p and p_1 and some final state with four-momenta p' and p'_1;

- $h_q\left[f_q, f_{q1}\right]$ = correlation function related to the presence of two particles in the same space-time position x but with different four-momenta p and p_1, respectively.

Remarks:

- $f_q^q = (f_q)^q$

- C_q implies a new q generalized version of the Boltzmann molecular chaos hypothesis, according to which,

$$h_q\left[f_q, f_{q1}\right] = \exp_q\left[\ln_q f_q + \ln_q f_{q1}\right] \tag{38}$$

$$\text{where} \quad \begin{cases} \ln_q(X) = \dfrac{X^{1-q} - 1}{1 - q} \\ \exp_q(X) = [1 + (1 - q)X]^{\frac{1}{1-q}} \end{cases} \tag{39}$$

The divergence of the entropy current [13] defined as,

$$s_q^\mu(x) \equiv -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \big\{ f_q^q(x,p) \ln_q f_q(x,p) - f_q(x,p) \big\} \tag{40}$$

is always positive at any space-time point,

$$\partial_\mu s_q^\mu(x) \geqslant 0 \tag{41}$$

In order to perform the calculations we rewrite the entropy current from equation 40 as,

$$\begin{aligned} s_q^\mu(x) &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \left\{ f_q^q(x,p)\, \frac{f_q^{q-1}(x,p) - 1}{1-q} - f_q(x,p) \right\} \\ &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \left\{ \frac{f_q(x,p)}{1-q} - \frac{f_q^q(x,p)}{1-q} - f_q(x,p) \right\} \end{aligned} \tag{42}$$

With this new form, we proceed to calculate the divergence of the entropy current.

$$\begin{aligned} \partial_\mu s_q^\mu(x) &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \left\{ \frac{\partial_\mu f_q(x,p)}{1-q} - \frac{q f_q^{q-1}(x,p) \partial_\mu f_q(x,p)}{1-q} - \partial_\mu f_q(x,p) \right\} \\ &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \left\{ \frac{1}{1-q} - \frac{q f_q^{q-1}(x,p)}{1-q} - 1 \right\} \partial_\mu f_q(x,p) \\ &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \left\{ \frac{q}{1-q} - \frac{q f_q^{q-1}(x,p)}{1-q} \right\} \partial_\mu f_q(x,p) \\ &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} q f_q^{q-1}(x,p)\, \underbrace{\frac{f_q^{1-q}(x,p) - 1}{1-q}}_{\ln_q f_q(x,p)} \partial_\mu f_q(x,p) \end{aligned} \tag{43}$$

We identify the definition of the q logarithm given in 39 and we obtain,

$$\begin{aligned} \partial_\mu s_q^\mu(x) &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \ln_q f_q(x,p)\, \underbrace{q f_q^{q-1}(x,p) \partial_\mu f_q(x,p)}_{\partial_\mu f_q^q(x,p)} \\ &= -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{p^\mu}{p^0} \ln_q f_q(x,p) \partial_\mu f_q^q(x,p) \end{aligned} \tag{44}$$

Making use of the non-extensive form of the BTE given in equation 36, we get,

$$\partial_\mu s_q^\mu(x) = -k_{\mathrm{B}} \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{p^0} \ln_q f_q(x,p) C_q(x,p) \tag{45}$$

Imposing the momentum conservation in two-particle collisions,

$$p^\mu + p_1^\mu = p'^\mu + p_1'^\mu \tag{46}$$

we form the collision invariant,

$$F[\psi] = \int \frac{d^3p}{p^0}\psi(x,p)C_q(x,p) \equiv 0 \tag{47}$$

where,

$$\psi(x,p) = a(x) + b_\mu(x)p^\mu \quad , \quad a(x), b_\mu(x) = \text{arbitrary functions} \tag{48}$$

We identify in equation 45 exactly the form of the collision invariant introduced above, with $\psi(x,p) = \ln_q f_q(x,p)$,

$$\partial_\mu s_q^\mu(x) = -k_\mathrm{B} \underbrace{\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{p^0}\ln_q f_q(x,p)C_q(x,p)}_{F[\ln_q f_q(x,p)]} \tag{49}$$

$$\implies \partial_\mu s_q^\mu(x) = -k_\mathrm{B}F[\ln_q f_q(x,p)] \tag{50}$$

Since $\psi(x,p) = \ln_q f_q(x,p)$ we get,

$$\begin{aligned} \ln_q f_q(x,p) &= a(x) + b_\mu(x)p^\mu \\ \frac{f_q^{1-q}(x,p) - 1}{1-q} &= a(x) + b_\mu(x)p^\mu \\ f_q^{1-q}(x,p) &= (1-q)\big[a(x) + b_\mu(x)p^\mu\big] + 1 \end{aligned} \tag{51}$$

$$\implies \boxed{f_q(x,p) = \Big[(1-q)\big[a(x) + b_\mu(x)p^\mu\big] + 1\Big]^{\frac{1}{1-q}}} \tag{52}$$

3.3 Non-Extensive BTE in the RTA

We saw in Sections 2.1 and 2.2 that one approximation which leads to analytical solutions of the Boltzmann transport equation is the relaxation time approximation. Even if we consider the simplifying assumption that our system is a homogeneous plasma with no external force, finding an exact analytical solution of the non-extensive BTE in the RTA is quite difficult and what we can do is to arrive at some approximate iterative analytical solutions [14].

The non-extensive BTE for the homogeneous distribution f with no external force in the relaxation time approximation is given by,

$$\begin{gathered} \frac{\partial f^q}{\partial t} = -\frac{f - f_\mathrm{eq}}{\tau} \implies qf^{q-1}\frac{\partial f}{\partial t} = -\frac{f - f_\mathrm{eq}}{\tau} \\ \frac{\partial f}{\partial t} = -\frac{f^{2-q} - f_\mathrm{eq}f^{1-q}}{q\tau} \end{gathered} \tag{53}$$

We separate the terms for integration,

$$-\frac{1}{q\tau}\int dt = \int \frac{df}{f^{2-q} - f_{\text{eq}}f^{1-q}} \tag{54}$$

and we denote $w \equiv f^{q-1}$ in order to rewrite the right hand side of the above equation as,

$$\frac{df}{f^{2-q} - f_{\text{eq}}f^{1-q}} = \frac{1}{q-1}\frac{dw}{1 - f_{\text{eq}}w^{-\frac{1}{q-1}}} \tag{55}$$

What we get is,

$$\kappa - \frac{t}{q\tau} = \frac{1}{q-1}\int \frac{dw}{1 - f_{\text{eq}}w^{-\frac{1}{q-1}}} \tag{56}$$

where κ is the integration constant which can be obtained from the boundary condition, $f(t=0) = f_{\text{in}}$, f_{in} being the initial distribution. Moreover, we denote $\theta = t/q\tau$.

It follows that the integral we want to solve becomes,

$$\boxed{\frac{1}{q-1}\int \frac{dw}{1 - f_{\text{eq}}w^{-\frac{1}{q-1}}} = \kappa - \theta} \tag{57}$$

Since $\left|f_{\text{eq}}w^{-\frac{1}{q-1}}\right| = |f_{\text{eq}}f^{-1}| = \left|\frac{f_{\text{eq}}}{f}\right| < 1$ we can expand the integrand in a negative binomial series and integrate,

$$\begin{aligned} \frac{1}{q-1}\int dw\left(1 + f_{\text{eq}}w^{-\frac{1}{q-1}} + f_{\text{eq}}^2 w^{-\frac{2}{q-1}} + \dots\right) &= \kappa - \theta \\ \implies \frac{f^{q-1}}{q-1}\sum_{s=0}^{\infty}\frac{(1)_s(1-q)_s}{s!(2-q)_s}\left(\frac{f_{\text{eq}}}{f}\right)^s &= \kappa - \theta \\ \implies \frac{f^{q-1}}{q-1}{}_2F_1\left(1, 1-q; 2-q; \frac{f_{\text{eq}}}{f}\right) &= \kappa - \theta \end{aligned} \tag{58}$$

where '$(.)_s$' is the rising Pochhamer symbol [23] defined as,

$$(a)_s = \begin{cases} 1 & , \quad s = 0 \\ a(a+1)\dots(a+s-1) & , \quad \forall s > 0 \end{cases}$$

and ${}_2F_1$ is the hypergeometric function [24].

In order to determine the integration constant we set $t = 0 \implies \theta = 0$ and we get,

$$\kappa = \frac{f_{\text{in}}^{q-1}}{q-1}{}_2F_1\left(1, 1-q; 2-q; \frac{f_{\text{eq}}}{f_{\text{in}}}\right) \tag{59}$$

In the relaxation time approximation, the solution of the non-extensive BTE can be derived by solving equation 58 for f. Although numerical methods can be used to find the solution to this equation, we will use the series expansion of the hypergeometric function given in equation 58 to calculate approximate analytical expressions for the solutions.

Zeroth order solution: $s = 0$

$$\frac{f^{q-1}}{q-1} = \kappa - \theta \implies f^{q-1} = \big(q-1\big)\big(\kappa - \theta\big) \tag{60}$$

$$\begin{aligned} &\implies \Psi_0 = f^{q-1} - \big(q-1\big)\big(\kappa-\theta\big) = 0 \\ &\implies f_0 = \big[\big(q-1\big)\big(\kappa-\theta\big)\big]^{\frac{1}{q-1}} \end{aligned} \tag{61}$$

First order equation, whose solution we denote by $f_1(t)$, is given by,

$$\frac{f^{q-1}}{q-1} + \frac{f_{\text{eq}} f^{q-2}}{q-2} = \kappa - \theta \implies f^{q-1} + \frac{q-1}{q-2} f_{\text{eq}} f^{q-2} = \big(q-1\big)\big(\kappa-\theta\big) \tag{62}$$

$$\Psi_1 = f^{q-1} + \frac{1-q}{2-q} f_{\text{eq}} f^{q-2} - \big(q-1\big)\big(\kappa-\theta\big) = 0 \tag{63}$$

Exact solution, which we denote by $f_{\text{e}}(t)$, is obtained from the following equation,

$$\Psi_{\text{e}} = \frac{f^{q-1}}{q-1} {}_2F_1\left(1, 1-q; 2-q; \frac{f_{\text{eq}}}{f}\right) - \kappa + \theta = 0 \tag{64}$$

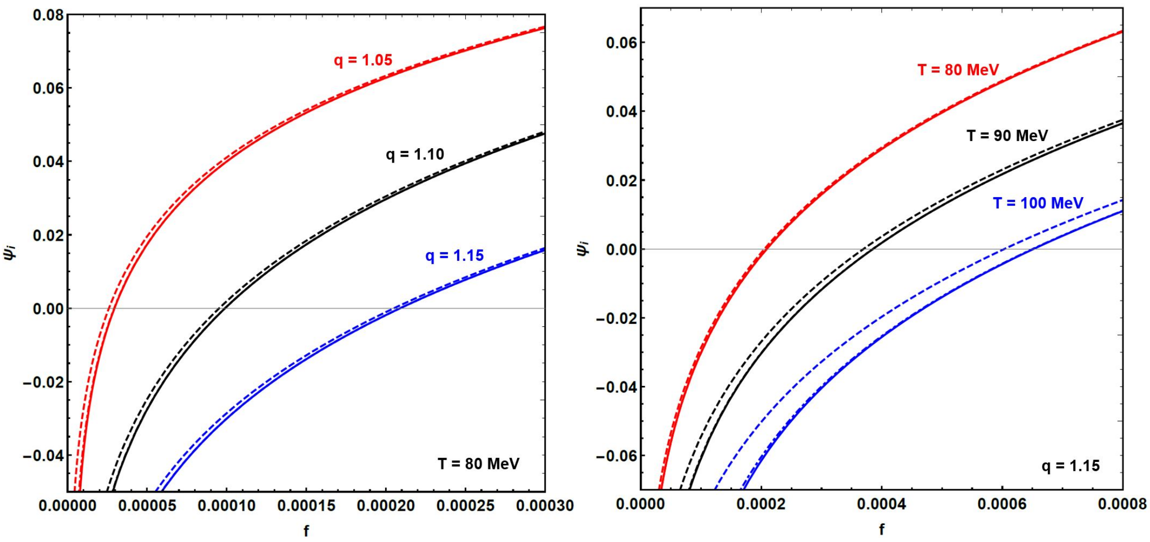
We saw that using the inverse function, it is simple to achieve the zeroth order solution. However, this becomes more complicated for higher order equations. An approximate analytical first order solution for the non-extensive BTE in the relaxation time approximation can be obtained based on the graphical solutions of equations 61, 63 and 64. These graphical solutions show us how close to each other are the solutions of equations 61, 63 and 64. The solution of the zeroth order equation is very close to that of the exact equation, while the solution of the first order equation almost entirely overlaps with the exact solution.

Thus, due to how close the solutions are to each other, we propose to write the solution of the first order equation as the solution of the zeroth order solution plus a small increment,

$$f_1 = f_0 + \epsilon_1, \quad |\epsilon_1| << f_0 \tag{65}$$

Putting equation 65 into equation 63, we get,

$$\Psi_1 = (f_0 + \epsilon_1)^{q-1} + \frac{1-q}{2-q} f_{\text{eq}} (f_0 + \epsilon_1)^{q-2} - \big(q-1\big)\big(\kappa-\theta\big) = 0 \tag{66}$$



(a) (b)

Figure 5: Graphical solutions of equations 61, 63 and 64 for different q values on the left (5.a) and for different T values on the right (5.b). The mass is $m = 139.57$ MeV, the transvers momentum is $p_T = 1$ GeV, and the product $q\theta = 0.11$. We represented the solution of the zeroth order equation with a dashed line, the solution of the first order equation with a dot-dashed line and that of the exact solution with a solid line. The figure is reproduced from Ref. [14].

Using Mathematica to expand in terms of ϵ_1 up to the first order, since ϵ_1 is a small quantity, we solve for ϵ_1 which has the following expression,

$$\epsilon_1 = \frac{f_0}{f_0 + f_{\rm eq}}\left(\frac{f_0}{1-q} + \frac{f_{\rm eq}}{2-q} + f_0^{2-q}(\kappa - \theta)\right) \tag{67}$$

Hence, we get the following expression for the solution of the first order equation,

$$f_1 \approx f_0 + \frac{f_0}{f_0 + f_{\rm eq}}\left(\frac{f_0}{1-q} + \frac{f_{\rm eq}}{2-q} + f_0^{2-q}(\kappa - \theta)\right) \tag{68}$$

Analogously, we write the solution of the second order equation as a small increment over that of the first order in the following way,

$$f_2 = f_1 + \epsilon_2, \quad |\epsilon_2| << f_1 \tag{69}$$

Using Mathematica again to expand in terms of ϵ_2 up to the first order, since ϵ_2 is a small quantity, we solve for ϵ_2 which has the following expression,

$$\epsilon_2 = \frac{f_1}{f_1^2 + f_{\rm eq}f_1 + f_{\rm eq}^2}\left(\frac{f_1^2}{1-q} + \frac{f_{\rm eq}f_1}{2-q} + \frac{f_{\rm eq}^3}{3-q} + f_1^{3-q}(\kappa - \theta)\right) \tag{70}$$

Hence, we get the following expression for the solution of the first order equation,

$$f_2 \approx f_1 + \frac{f_1}{f_1^2 + f_{\text{eq}} f_1 + f_{\text{eq}}^2} \left(\frac{f_1^2}{1-q} + \frac{f_{\text{eq}} f_1}{2-q} + \frac{f_{\text{eq}}^3}{3-q} + f_1^{3-q}(\kappa - \theta) \right) \tag{71}$$

Generalizing, the first order and the higher order solutions can be represented as,

$$\boxed{f_i = f_{i-1} + \epsilon_i} \quad , \quad i = 1, 2, 3... \tag{72}$$

where, ϵ_i can be calculated from the following expression,

$$\boxed{\epsilon_i = \frac{f_{i-1}}{\sum_{r=0}^{i} f_{\text{eq}}^r f_{i-1}^{i-r}} \left(f_{i-1}^{i+1-q}(\kappa - \theta) + \sum_{r=0}^{i} \frac{f_{\text{eq}}^r f_{i-1}^{i-r}}{r+1-q} \right)} \tag{73}$$

4 Summary and Conclusions

To conclude, we reviewed some important aspects of QGP, we studied the evolution of energetic particles inside QGP which are effective probes in the study of the properties of the plasma medium and we connected our findings to the experimental observables. We started by describing the evolution of the phase space distribution functions of the high-energy particles in the framework of the Boltzmann-Gibbs statistics, deriving the Boltzmann transport equation (BTE). For a uniform QGP in the absence of any external force, we solved analytically the BTE in the relaxation time approximation (RTA). Passing then to the experimental observables, we discussed about the transverse momentum (p_T) distribution of the hadrons produced in proton-proton and heavy-ion collisions. These particle spectra are very important and they allow us to determine other experimental observables such as the nuclear suppression factor (R_{AA}) and the elliptic flow. We defined R_{AA} as the ratio of the final distribution to the initial distribution parameterized by a Tsallis-like function. We plotted R_{AA} as a function of p_T for different values of the non-extensivity parameter q and for different values of the temperature T. We also briefly introduced the elliptic flow since it is an experimental observable which is sensitive to the early evolution of the plasma medium. After that, we discussed about the transport of high-energy particles inside non-extensive QGP. Since experimental data are well fitted by the Tsallis-like power-law distribution function, we derived the Tsallis probabilities starting from entropy. Having the probabilities, it is easy to get the distribution functions of the energetic particles. Since high-energy collisions yield a power-law stationary state instead of an exponential thermal equilibrium, we studied a modified BTE inspired by the Tsallis statistics that yields such a solution. Finding analytical solutions of the non-extensive BTE is a hard task and we proposed a simpler path by using the RTA for a uniform QGP with no external force, as we did in Section 2, in order to apply a perturbative approach which led us to some approximate iterative solutions for the distribution functions [14]. Hence, the generalized Tsallis statistics is a good theoretical framework for treating the non-extensive features of QGP, describing remarkably well the experimental data coming from high-energy collisions at RHIC and LHC and opening the path for a better understanding of the properties of QGP.

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