Numerical Methods in the theory of Topological Solitons

– Student Report –

submitted by

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November 2020

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JINR INTEREST PROGRAM 2020

Abstract

In this report we discuss the kink and anti-kink solutions of systems like ϕ^4 theory , Sine Gordon model and ϕ^6 theory and finally we'll use numerical methods to obtain the kink and anti-kink solution for a certain problem where the potential interpolates between Sine Gordon, ϕ^4 and potential with 2 saddle points.

Acknowledgments

I would like to thank and express my sincere gratitude to my supervisor Dr.Yakov M.Shnir for his patient guidance and brilliant notes.
I would also like to to thank JINR committee for their online program INTEREST for providing such great chances in these hard times , it was a great idea . Thank you for the opportunity!
and very special thanks to my Colleague Sohair ELMeligy and Mr.Ahmed M.M.Gharib for their incredible support and help . I can't thank them enough!
Thank you all! it was invaluable experience

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1 Introduction

1.1 What are solitons?

Solitons are stable, localized configurations, particle-like wave packet that maintains its shape while it propagates at a constant velocity. The ending "-on" indicates the particle-like nature of the solution
One of the simplest examples of solitons is the class of the kink configurations, which appears in the (1+1) dimensional models with a potential possessing two or more degenerated minima.
To obtain the kink solution one needs to solve the corresponding field equations either analytically or numerically.

1.2 Obtaining a solution

for a linear scalar field ϕ, the Lagrangian can be written as

$$l = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$

From this Lagrangian we can now derive the corresponding field equation using Euler-Lagrange

$$\frac{\partial l}{\partial\phi} = \partial_\mu\frac{\partial l}{\partial(\partial_\mu\phi)}$$

$\frac{\partial l}{\partial\phi} = -\frac{dV}{d\phi}$

$\frac{\partial l}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$

$\partial_\mu\frac{\partial l}{\partial(\partial_\mu\phi)} = \partial^\mu\phi\partial_\mu\phi$

$$\partial^\mu\phi\partial_\mu\phi = -\frac{dV}{d\phi}$$

since our study would be on the 1+1 dimension ,$g_{\mu\nu} = diag(1,-1)$, the Lagrangian is

$$l = \frac{1}{2}\partial_t^2\phi - \frac{1}{2}\partial_x^2\phi - V(\phi)$$

and equation of motion is

$$\partial_t^2\phi - \partial_x^2\phi + \frac{dV}{d\phi} = 0$$

our interest is to study the kink and anti-kink solution ,so we will study the equation in the static frame where there's no time evolution.

note that the Lagrangian is Lorentz invariant so we can obtain the solution in the static frame and then boost it to get the moving solution

so $\partial_t^2\phi = 0$

$$l = -\frac{1}{2}\partial_x^2\phi - V(\phi) \tag{1}$$

$$\partial_x^2\phi - \frac{dV}{d\phi} = 0 \tag{2}$$

taking x- derivative of

$$\frac{d}{dx}\left(\frac{1}{2}\partial_x^2\phi - V(\phi)\right) = 2.\frac{1}{2}d_x\phi d_x^2\phi - \frac{dV}{d\phi}.\frac{d\phi}{dx} = 0$$

$$d_x\phi\left(d_x^2\phi - \frac{dV}{d\phi}\right) = 0$$

we notice that the term in the brackets is the same as the L.H.S from equation (2). since we know from the equation of motion that this term vanishes, then we have:

$$\frac{1}{2}\partial_x^2\phi - V(\phi) = const$$

now we need to find this constant. since we're interested in the finite energy solution , then $\frac{\partial\phi}{\partial x} \xrightarrow{|x|\to\infty} 0$

$V(\phi) \xrightarrow{|x|\to\infty} 0$

applying these conditions to equation, we evaluate the constant to equal 0

$$\frac{1}{2}(d_x\phi)^2 = V(\phi)$$

taking the root

$$\frac{d\phi}{dx} = \pm\sqrt{2V(\phi)} \tag{3}$$

using this equation (3) , we'll substitute with each potential (V) in every model and integrate both sides to get the static solution

$$\int dx = \pm\int\frac{d\phi}{\sqrt{2V}}$$

2 Sine Gordon Model

the potential in Sine Gordon model is:

$$V(\phi) = 1 - Cos\phi$$

[3] This potential has an infinite number of degenerate vacua at $\phi_o = 2\pi n, n \in \mathbb{Z}$, for which $V''(\phi_o) = 1$
2 vacua at x=0 , x= 2π

$\int dx = \pm \int \frac{d\phi}{\sqrt{2}\sqrt{1-Cos\phi}}$

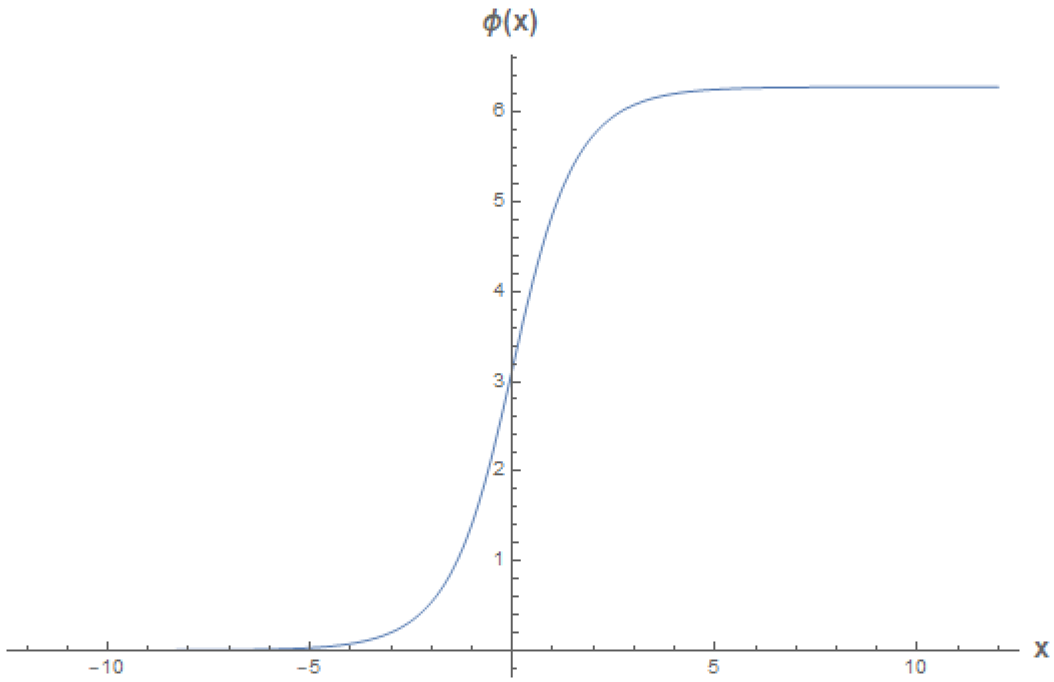
$\sqrt{1 - Cos\phi} = \sqrt{2} Sin\frac{\phi}{2}$

$\int dx = \pm \int \frac{d\phi}{2Sin\frac{\phi}{2}}$

$\int dx = \pm \int \frac{Sin^2\frac{\phi}{4}+Cos^2\frac{\phi}{4}}{4Sin\frac{\phi}{4}Cos\frac{\phi}{4}}$

$x - x_o = \pm \left(ln(Sin\frac{\phi}{4}) - ln(Cos\frac{\phi}{4}) \right)$
$x - x_o = \pm ln\left(tan\frac{\phi}{4} \right)$

$$\phi = 4tan^{-1}e^{(x-x_o)}$$



3 ϕ^4 theory

in the ϕ^4 theory the potential is

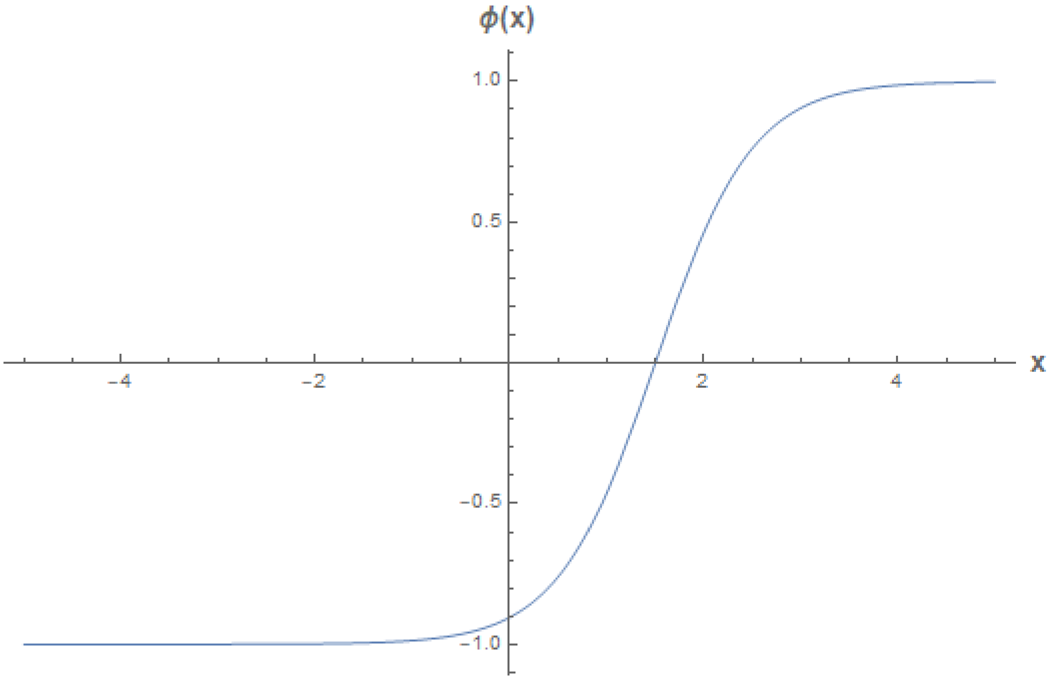
$$V(\phi) = \frac{1}{2}(1 - \phi^2)^2$$

it has 2 vacua at $\phi = 1, \phi = -1$

$\int dx = \pm \int \frac{d\phi}{\sqrt{(1-\phi)^2}} = \pm \int \frac{d\phi}{\sqrt{(1-\phi)}}$
$x - x_o = \pm arctanh\phi$

$$\phi = \pm tanh(x - x_o)$$

$$localizedatx_o = 1.5$$



4 ϕ^6 theory

in ϕ^6 theory the potential has the form:

$$V(\phi) = \frac{1}{2}\phi^2(1 - \phi^2)^2$$

and has 3 vacuas at 0, 1, -1
$\int dx = \int \pm \frac{d\phi}{\sqrt{2\frac{1}{2}\phi^2(1-\phi^2)^2}}$

$\int dx = \pm \int \left(\frac{1}{\phi} - \frac{0.5}{1-\phi} - \frac{0.5}{\phi+1}\right) d\phi$

$x - x_0 = \pm \left(ln\phi - \frac{1}{2}\left(ln(1-\phi) + ln(\phi + 1)\right)\right)$

$x - x_0 = \pm ln \frac{\phi}{\sqrt{(1-\phi^2)}}$

$e^{2(x-x_o)} = \pm\frac{\phi^2}{1-\phi^2}$

$$\phi_{(0,1)} = \sqrt{\frac{1}{1+e^{-2(x-x_o)}}} = \sqrt{\frac{1+tanh(x-x_0)}{2}}$$
$$\phi_{(-1,0)} = -\sqrt{\frac{1}{1+e^{2(x-x_o)}}} = -\sqrt{\frac{1-tanh(x-x_0)}{2}}$$

5 The model

We have 1+1 dimension with the potential

$$V(\phi) = (1-\epsilon)(1-Cos\phi) + \frac{\epsilon\phi^2}{8\pi^2}(\phi-2\pi)^2, \epsilon \in [0, 2.7]$$

at $\epsilon = 0$ it's the standard Sine Gordon potential, at $\epsilon = 1$ it's shifted and rescaled $\phi 4$ model
, at $\epsilon = 2.7$ it's potential with 2 saddle points

$$\frac{dv}{d\phi} = (1-\epsilon)Sin\phi + \frac{\epsilon\phi}{2\pi^2}(\phi-2\pi)(\phi-\pi)$$

from equation (1) the corresponding equation of motion is

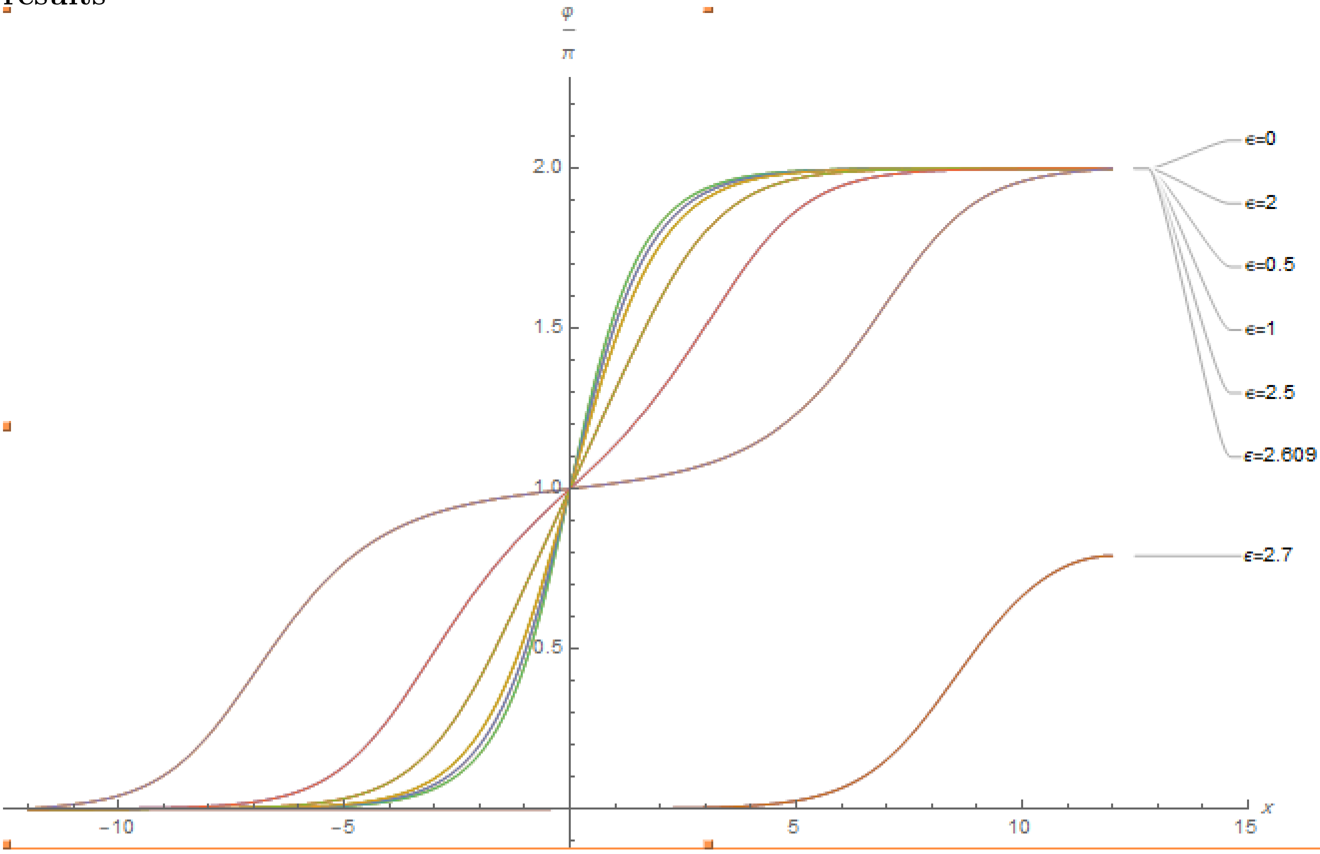
$$\phi_{tt} - \phi_{xx} + (1-\epsilon)sin\phi + \frac{\epsilon\phi}{2\pi^2}(\phi-2\pi)(\phi-\pi) = 0$$

since we're in interested in finding the kink and anti kink solution the equation of motion in the static frame, the field equation is

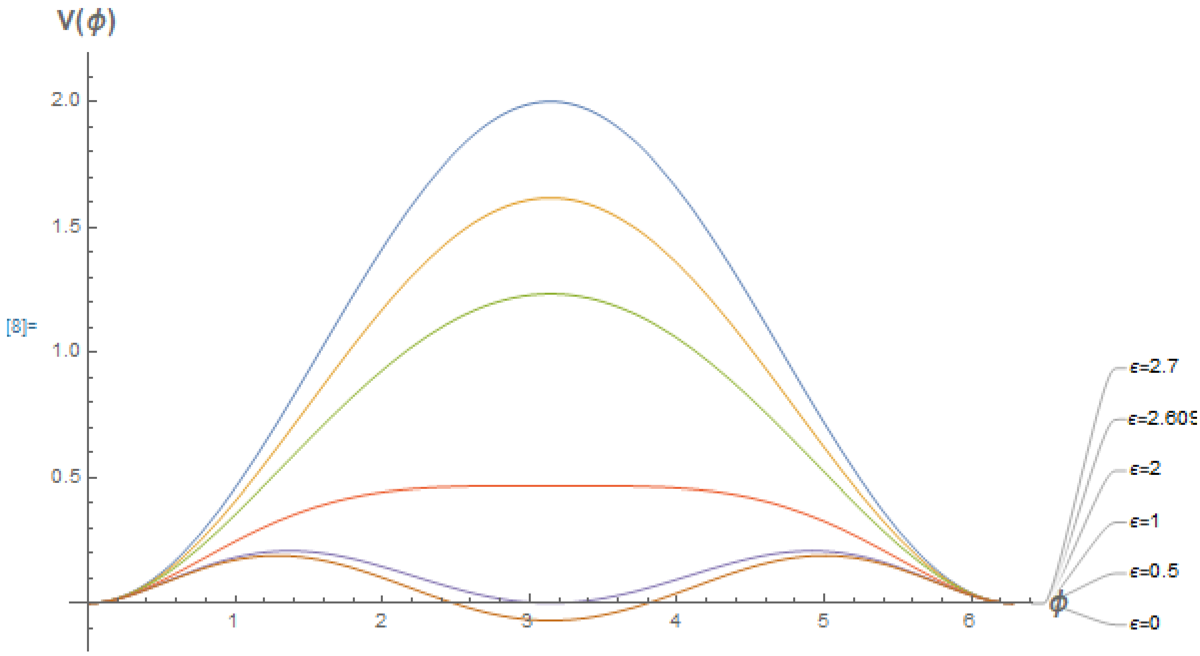
$$\phi_{xx} = (1-\epsilon)sin\phi + \frac{\epsilon\phi}{2\pi^2}(\phi-2\pi)(\phi-\pi)$$

with the boundary conditions:
$\phi(-\infty) = 0$,
$\phi(+\infty) = 2\pi$
This is a boundary value problem , we will solve it numerically using the shooting method

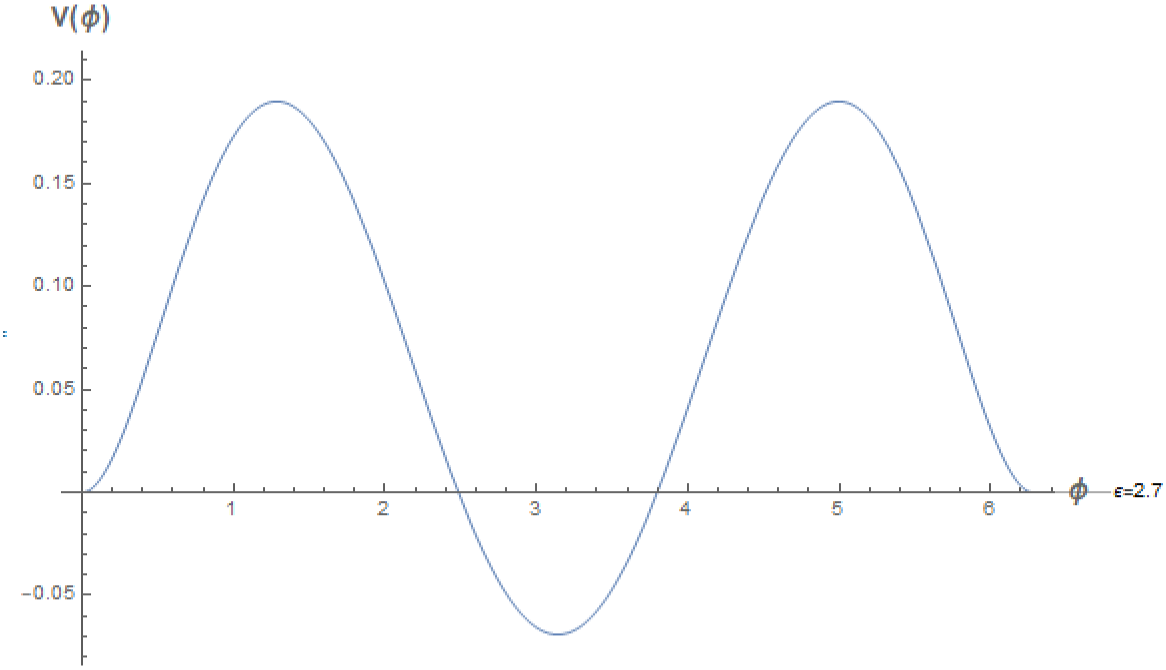
results



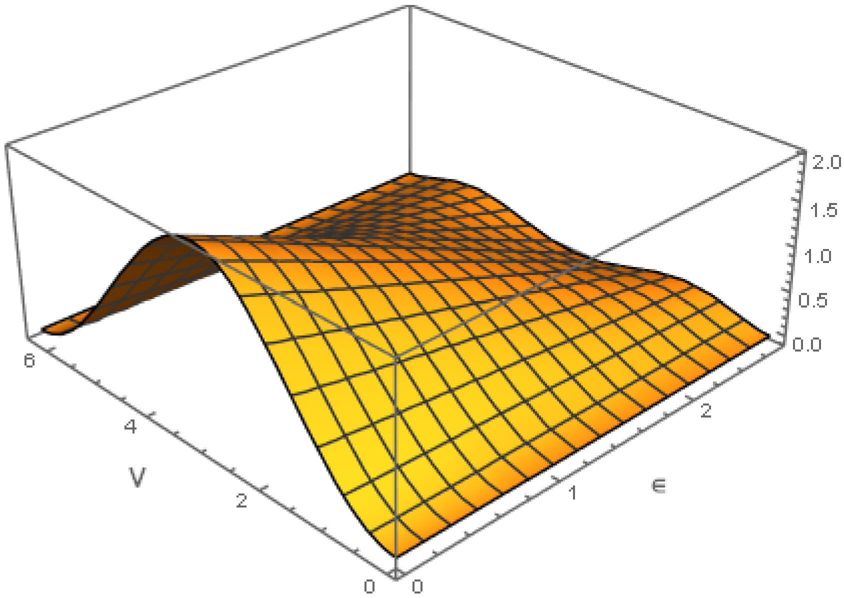
potential shape:

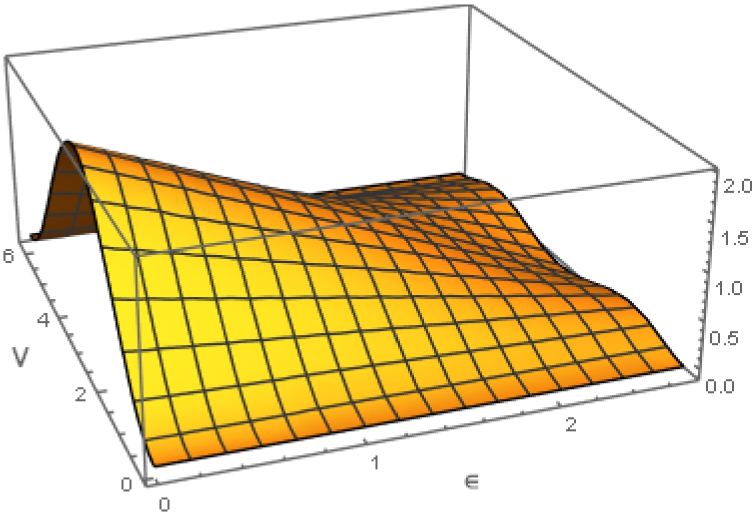


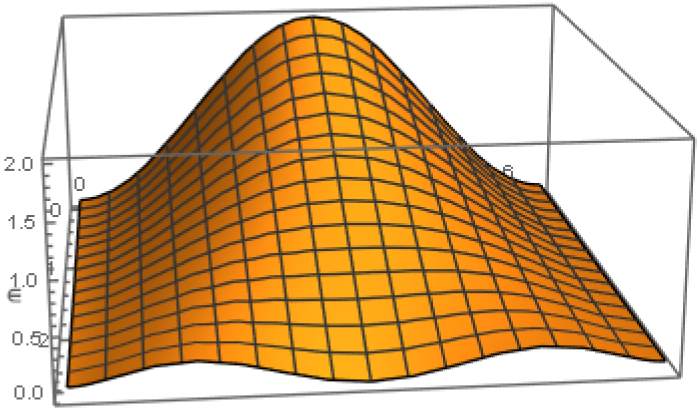
note the 2 saddle points at 2.7



The potential varying with the parameter ϵ







6 Conclusion

After deriving the kink,anti-kink solution analytically in 3 well-known models ,e.g., Sine Gordon, ϕ^4 and ϕ^6 theory.

We moved to a different problem where the parameter varying potential interpolates between 3 cases, A solution was obtained numerically using the shooting method and we saw how the solution varied with the parameter ϵ

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