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Non-linear Waves in Extensive and Non-extensive Quark-Gluon Plasma

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Abstract

The collisions of heavy ions are performed to study the quark-gluon plasma and the beginning of the universe. In this report, the energy density and pressure of quark-gluon plasma were calculated depending on Maxwell-Boltzmann distribution function. In addition, propagation of nonlinear waves inside the QGP medium has been studied considering the background medium as the one following the Boltzmann-Gibbs and Tsallis statistics.

Acknowledgment

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1 Introduction

Experiments such as STAR and PHENIX at Brookhaven National Laboratory and ALICE and ATLAS at CERN were conducted by colliding heavy-ions like Au/Pb to create the hot and dense Quark Gluon Plasma (QGP) matter and study its properties. Studying the Quark-Gluon Plasma (QGP) could be possible through the internal probes such as high-energetic particles, which are generated in the pre-equilibrium phase. These particles generate nonlinear waves while passing through QGP and these waves evolve with time [3, 4]. The reason for using such an internal probe is due to the fact that the QGP medium lasts for a very short time. Tsallis statistics [5] is a new form of statistics, which is generalized to study the non-extensive quark-gluon plasma [6] and it follows the power law distribution (i.e. $E^{-\alpha}$). In contrast, Boltzmann-Gibbs statistics is employed to study the extensive type of the QGP and described by the exponential distribution (i.e. $e^{\frac{-E}{T}}$).

1.1 Quark-Gluon Plasma

The term "Quark-Gluon plasma" means that we have a plasma of quarks and gluons (i.e. deconfinement phase), which are called "Partons". This state of matter could be possible when two heavy-ions collided at $\sqrt{s} = 10 - 15$ GeV per nucleon [7, 8]. However, it is difficult to separate a single quark [2]. Ultra-relativistic heavy ion collisions aim to investigate many aspects of particle physics, most importantly, simulate and understand the formation of universe. Quark-gluon plasma could be obtained in the laboratory system through two ways:

- **High Temperature**: Reaching an extreme value of temperature about 170 MeV can be possible by colliding two heavy nuclei at high energies. At this temperature, hadrons are melted to form the QGP [9] and it will be formed within 1 fm/c after the collisions of the two beams, while the hadronization starts after about 10 fm/c. Figure 1 [1] shows the QGP evolution of the space-time, where T_{fo}, T_{ch} and T_c are the temperatures when hadrons stop elastic collisions, when hadrons have their stable form and stop inelastic collisions with themselves and when QGP is about to transfer to the hadronization phase [8].

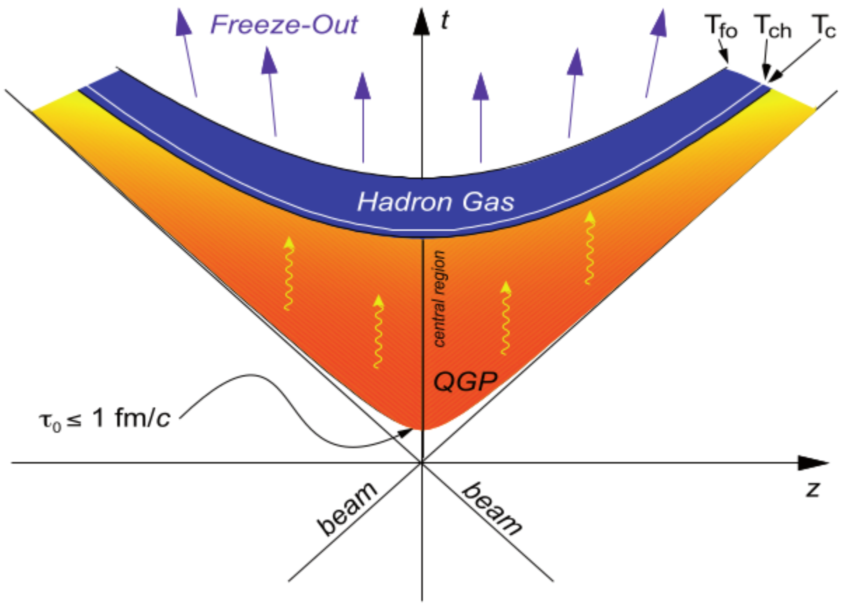
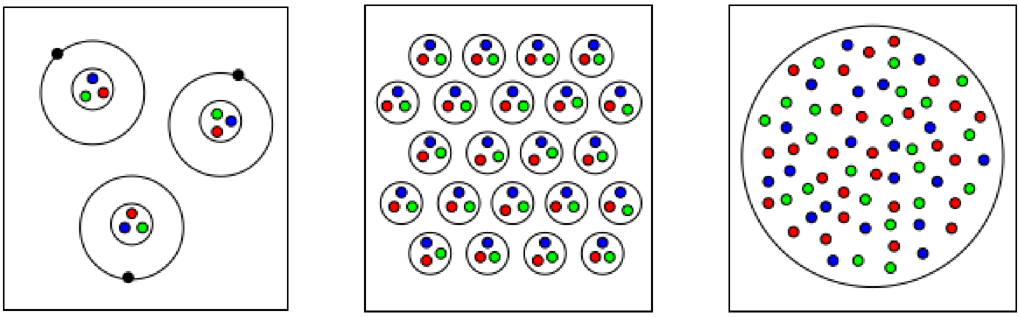


Figure 1: Evolution of heavy-ion collisions in the space-time frame [1]

- **High Pressure**: As presented in figure 2 [2], we start from the atomic scale of matter, after stripping the electrons, we will have hadrons. By increasing the density of the participating hadrons and extremely pressurizing them, this will leads to formation of the QGP state of matter.



(a) Atomic Scale (b) Hadronic Scale (c) QGP Formation

Figure 2: Transfer from the atomic scale to the QGP at higher densities (compressing) [2]

1.2 Propagation of Waves in QGP

Studying the effect of propagated high energetic particles, which are produced in the pre-equilibrium phase [4] inside the QGP acts as a promising technique to understand the its properties, which could be done by analyzing the perturbations resulting from deposition of particle's energy. These perturbations in the system could be linear or non-linear. Considering the shape of the waves resulted, we have two solutions:

- **Soliton solution:** This is the solution of the Korteweg-de Vries equation (KdV), which depends on the equations of hydrodynamics and this solution appears when the wave maintains its shape without breaking [3].
- **Breaking Wave Solution:** This arises when the waves do not maintain the initial shape as a function of time and break [4].

2 Hydrodynamics and Mathematical Formalism

2.1 Relativistic Hydrodynamics and QGP

To describe the evolution of space-time, relativistic hydrodynamics is used to study energy density and the fluctuations in fluids such as the QGP [3, 4]. We can get the equations of hydrodynamics due to conservation of the energy and momentum in a gas [10].

2.2 Relativistic Euler's Equation

Let us consider the pressure as P and energy density as ϵ. We assume there is a frame of reference in fluid "at rest" at specific space and time.
In an ideal fluid, the energy-momentum tensor is of the form [11]:

$$T_0 = \begin{pmatrix} \epsilon & 0 \\ 0 & P\delta_{ij} \end{pmatrix} \tag{2.1}$$

The energy-momentum tensor [11, 12] of a fluid that moves with velocity v is given as:

$$T_{\mu\nu} = P\eta_{\mu\nu} + (\epsilon + P)u_\mu u_\nu \tag{2.2}$$

For a fluid, the conservation of energy and momentum [12] will give:

$$\partial_\mu T_{\mu\nu} = \partial_\nu P + \partial_\mu \left[(\epsilon + P)u_\mu u_\nu\right] = 0 \tag{2.3}$$

Equation (2.3) vanishes (i.e. $\partial_\nu T_{\mu\nu} = 0$) and leads to the following two equations [11]:

- At $\nu = 0$, eq. (2.3) becomes:

$$\frac{\partial P}{\partial t} - \frac{\partial}{\partial t}\left\{\frac{\epsilon + P}{1 - v^2}\right\} + \nabla.\left\{\frac{(\epsilon + P)v}{1 - v^2}\right\} = 0$$

- While, at $\nu = i$, we have:

$$\frac{\partial v_i}{\partial t} + (v.\nabla)v_i = \frac{(v^2 - 1)}{(\epsilon + P)}\left(\nabla_i P + v_i \frac{\partial P}{\partial t}\right) \tag{2.4}$$

From eq. (2.4), we will get equation 2.6, representing the one dimensional (i.e. x-direction) form of the Euler's relativistic equation [13]

$$\boxed{\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{(v^2 - 1)}{(\epsilon + P)}\left(\frac{\partial P}{\partial x} + v\frac{\partial P}{\partial t}\right)} \tag{2.5}$$

2.2.1 Derivation of Continuity Equation for Entropy Density (s)

Also, from the relativistic form of the continuity equation for the density of baryons [3, 12]:

$$\partial_\nu {j_B}^\nu = 0 \tag{2.6}$$

Where $\nu {j_B}^\nu = u^\nu \rho_B$, then eq. (2.6) can be written as:

$$\frac{\partial \rho_B}{\partial t} + \frac{v\rho_B}{(1 - v^2)}\left(\frac{\partial v}{\partial t} + \vec{v}.\vec{\nabla} v\right) + \vec{\nabla}.\left(\rho_B \vec{v}\right) = 0 \tag{2.7}$$

The projection of eq. (2.3) on the u^ν direction gives the density of entropy continuity equation in its relativistic form [13]:

$$(\epsilon + P)\partial_\mu u^\mu + u^\mu \partial_\mu \epsilon = 0 \tag{2.8}$$

By using Gibbs relation:

$$\epsilon + P = \mu_B \rho_B + Ts \tag{2.9}$$

From the first thermodynamics law:

$$d\epsilon = Tds + \mu_B d\rho_B \tag{2.10}$$

Using the conditions that the baryon density is zero, so $\rho_B = d\rho_B = 0$ and the temperature is high as we have a hot quarks-gluons gas $T \neq 0$, then eq. (2.10) becomes:

$$\epsilon = Ts \tag{2.11}$$

By substituting with equations (2.9) and (2.11) in eq. (2.8) we get:

$$(\mu_B \rho_B + Ts)\partial_\mu u^\mu + u^\mu \partial_\mu Ts = 0 \quad\Longrightarrow\quad Ts\partial_\mu u^\mu + u^\mu \partial_\mu Ts = 0 \tag{2.12}$$

Finally, for an ideal fluid we estimate that:

$$\partial_\nu (su^\nu) = 0 \tag{2.13}$$

The last equation will then expanded to get:

$$\boxed{\frac{\partial s}{\partial t} + \frac{1}{1 - v^2} vs \left(\frac{\partial v}{\partial t} + \vec{v}.\vec{\nabla} v\right) + \vec{\nabla}.(s.\vec{v}) = 0} \tag{2.14}$$

3 Energy Density and Pressure in the QGP Using Maxwell-Boltzmann Distribution Function

The description of the system's dynamics depends on identification of the state variables such as energy density(ϵ) and pressure (p). The Maxwell-Boltzmann distribution function is given in terms of energy E, chemical potential $\mu = 0$ and temperature T by the following form [14]:

$$f(p) = \frac{1}{(2\pi^3)} e^{(\frac{\mu - E}{T})} = \frac{1}{(2\pi^3)} e^{(\frac{-E}{T})} \tag{3.1}$$

In Boltzmann's distribution, the energy density is given by:

$$\epsilon = \frac{1}{(2\pi^3)} \int E e^{(\frac{-E}{T})} d^3p \tag{3.2}$$

On the other hand, the pressure is given by the following form:

$$p = \frac{1}{(2\pi^3)} \int \frac{|\vec{p}|^2}{3E} e^{(\frac{-E}{T})} d^3p \tag{3.3}$$

Defining dimensionless variables in terms of mass (m), temperature (T) and momentum ($\vec{p}$), we have:

$$\frac{m}{T} = z \quad \text{and} \quad \frac{E}{T} = \tau$$

The energy is given by:

$$E = \sqrt{p^2 + m^2} = T\tau$$

$$|p| = T(\tau^2 - z^2)^{\frac{1}{2}} \qquad |\vec{p}| d|\vec{p}| = T^2 \tau d\tau \qquad |\vec{p}|^2 d|p| = T^3 \tau (\tau^2 - z^2)^{\frac{1}{2}} d\tau$$

From modified Bessel function of second kind:

$$K_n(z) = \frac{2^n n!}{2n!} \frac{1}{z^n} \int_z^\infty (\tau^2 - z^2)^{n - \frac{1}{2}} e^{-\tau} d\tau \tag{3.4}$$

To get the first and second orders of K, we substitute by 1 and 2 instead of n

$$K_1(z) = \frac{2^1 1!}{2!} \frac{1}{z^1} \int_z^\infty (\tau^2 - z^2)^{1 - \frac{1}{2}} e^{-\tau} d\tau = \frac{1}{z} \int_z^\infty (\tau^2 - z^2)^{\frac{1}{2}} e^{-\tau} d\tau \tag{3.5}$$

$$K_2(z) = \frac{2^2 2!}{4!} \frac{1}{z^2} \int_z^\infty (\tau^2 - z^2)^{2 - \frac{1}{2}} e^{-\tau} d\tau = \frac{1}{3z^2} \int_z^\infty (\tau^2 - z^2)^{\frac{3}{2}} e^{-\tau} d\tau \tag{3.6}$$

3.1 Energy Density

The energy density is given by:

$$\epsilon = \frac{1}{(2\pi)^3} \int_z^\infty E e^{(\frac{-E}{T})} d^3p \tag{3.7}$$

Then, by changing the variables:

$$\epsilon = \frac{T^4}{2\pi^2} \int_z^\infty \tau^2 (\tau^2 - z^2)^{\frac{1}{2}} e^{-\tau} d\tau$$

By adding and subtracting z^2 inside the integral

$$\epsilon = \frac{T^4}{2\pi^2}\int_z^\infty [\tau^2 - z^2 + z^2](\tau^2 - z^2)^{\frac{1}{2}} e^{-\tau} d\tau \tag{3.8}$$

By redistributing the right-hand bracket to the left-hand one inside the integral, we get:

$$\int_z^\infty (\tau^2 - z^2)(\tau^2 - z^2)^{\frac{1}{2}} e^{-\tau} d\tau = \int_z^\infty (\tau^2 - z^2)^{\frac{3}{2}} e^{-\tau} d\tau \tag{3.9}$$

From equations (3.5) and (3.9)

$$\int_z^\infty (\tau^2 - z^2)^{\frac{3}{2}} e^{-\tau} d\tau = 3z^2 K_2(z) \tag{3.10}$$

Similarly,

$$\int_z^\infty z^2(\tau^2 - z^2)^{\frac{1}{2}} e^{-\tau} d\tau = z^2 \int_z^\infty (\tau^2 - z^2)^{\frac{1}{2}} e^{-\tau} d\tau \tag{3.11}$$

From eq. (3.6) and eq.(3.11), we have

$$\int_z^\infty (\tau^2 - z^2)^{\frac{1}{2}} e^{-\tau} d\tau = zK_1(z) \tag{3.12}$$

By substituting from equations (3.10) and (3.12), eq. (3.8) will be on the form:

$$\epsilon = \frac{T^4}{2\pi^2}\left[3z^2 K_2(z) + z^3 K_1(z)\right] \tag{3.13}$$

$$\boxed{\epsilon = \frac{T^4}{2\pi^2}\left[3(\frac{m}{T})^2 K_2(\frac{m}{T}) + (\frac{m}{T})^3 K_1(\frac{m}{T})\right]} \tag{3.14}$$

3.2 Pressure

The pressure (P) is given by:

$$P = \frac{1}{(2\pi)^2}\int \frac{1}{3}\frac{|\vec{p}|^2}{E} e^{(\frac{-E}{T})} d^3p \tag{3.15}$$

Using

$$E = T\tau \quad \text{and} \quad |\vec{p}|^2 = T^2(\tau^2 - z^2)$$

$$P = \frac{T^4}{6\pi^2}\int_z^\infty (\tau^2 - z^2)^{\frac{3}{2}} e^{-\tau} d\tau \tag{3.16}$$

From equations (3.6) and (3.16), the integration will be valued as:

$$\int_z^\infty (\tau^2 - z^2)^{\frac{3}{2}} e^{-\tau} d\tau = 3z^2 K_2(z)$$

Finally, the pressure will be as the following:

$$P = \frac{T^4}{6\pi^2}(3z^2)K_2(z) = \frac{T^4}{2\pi^2}(z^2)K_2(z) \tag{3.17}$$

$$\boxed{P = \frac{T^4}{2\pi^2}(\frac{m}{T})^2 K_2(\frac{m}{T})} \tag{3.18}$$

Energy density and pressure help us to build the MIT bag equation of state [3], that will be used in the next section.

4 The Wave Equation at Finite Temperature T≠0 (Boltzmann Statistics)

Euler and continuity equations are combined [3].
Euler's equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})v = \frac{(v^2-1)}{(\epsilon+P)}\left(\vec{\nabla}P + \vec{v}\frac{\partial P}{\partial t}\right) \tag{4.1}$$

The continuity equation for the entropy density:

$$\frac{\partial s}{\partial t} + \frac{1}{(v^2-1)}vs\left(\frac{\partial v}{\partial t} + \vec{v}.\vec{\nabla}v\right) + \vec{\nabla}.(s\vec{v}) = 0 \tag{4.2}$$

As the and the baryon chemical potential vanishes then we have $\rho_B = 0$ and $\mu = 0$
According to the MIT bag model:

$$3(P+\mathcal{B}) = \epsilon - \mathcal{B} = \frac{8\pi^2}{15}T^4 + \frac{6}{\pi^2}\int_0^\infty \left[\frac{2}{(1+e^{\frac{p}{T}})}p^3 dp\right] \tag{4.3}$$

As $\mathcal{B}$ is the bag constant, so we have $\frac{\partial \mathcal{B}}{\partial T} = 0$, as the entropy is given by $s = (\frac{\partial p}{\partial T})$ at constant volume V

$$s = \frac{\partial}{\partial T}\left(-\mathcal{B} + \frac{37}{90}\pi^2 T^4\right) = 0 + 4\frac{37}{90}\pi^2 T^3 = 4\frac{37}{90}\pi^2 T^3 \tag{4.4}$$

By inserting $\mathcal{B} = \frac{37}{30}\pi^2(T_B)^4$ into $\epsilon - \mathcal{B} = \frac{37}{30}\pi^2(T)^4$, where ${T_B}^4$ is a suitable temperature number to get value of $\mathcal{B}$, so we have:

$$\epsilon = \frac{37}{30}\pi^2(T)^4 + \mathcal{B} = \frac{37}{30}\pi^2(T)^4 + \frac{37}{30}\pi^2(T_B)^4 = \frac{37}{30}\pi^2\left(T^4 + {T_B}^4\right) \tag{4.5}$$

By solving $\epsilon - \mathcal{B} = \frac{37}{30}\pi^2 T^4$ to get the temperature, then $T^4 = \frac{30}{37\pi^2}(\epsilon - \mathcal{B})$. The temperature is given by:

$$T = \left[\frac{30}{37\pi^2}(\epsilon - \mathcal{B})\right]^{\frac{1}{4}} \tag{4.6}$$

By inserting the previous equation into eq. (4.4), then we get the entropy:

$$s = 4\frac{37}{90}\pi^2\left[\frac{30}{37\pi^2}(\epsilon - \mathcal{B})\right]^{\frac{3}{4}} \tag{4.7}$$

Substituting by eq.(4.7) in the following equation:

$$\frac{\partial s}{\partial t} + \frac{1}{(v^2-1)}vs\left(\frac{\partial v}{\partial t} + \vec{v}.\vec{\nabla}v\right) + \nabla.(s\vec{v}) = 0 \tag{4.8}$$

We have:

$$(1-v^2)\left[\left(\frac{90}{148\pi^2 T^4}\right)\frac{\partial \epsilon}{\partial t} + \frac{\partial v}{\partial x} + \left(\frac{90v}{148\pi^2 T^4}\right)\frac{\partial \epsilon}{\partial x}\right] + v\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = 0 \tag{4.9}$$

From equation:

$$3(p+\mathcal{B}) = \epsilon - \mathcal{B} = \frac{37}{30}\pi^2 T^4 \tag{4.10}$$

$$\epsilon + p = \frac{148}{90}\pi^2 T^4 \tag{4.11}$$

Inserting eq.(4.11) into eq. (4.1), and by using $\vec{\nabla} P = \frac{1}{3}\vec{\nabla}\epsilon$ and also $\frac{\partial P}{\partial t} = \frac{1}{3}\frac{\partial \epsilon}{\partial t}$, so:

$$\frac{148}{30}\pi^2 T^4 \left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right] = (v^2-1)\left(\frac{\partial \epsilon}{\partial x} + v\frac{\partial \epsilon}{\partial t}\right) \tag{4.12}$$

We then rewrite equations (4.11) and (4.12) in terms of dimensionless variables:

$$\hat{\epsilon} = \frac{\epsilon}{\epsilon_0} \qquad \hat{v} = \frac{v}{c_s}$$

Where ϵ_0 is the reference energy density and then by expanding the previous equations in power of σ:

$$\hat{\epsilon} = 1 + \sigma\epsilon_1 + \sigma^2\epsilon_2 + ... \qquad \hat{v} = 1 + \sigma v_1 + \sigma^2 v_2 + ...$$

Changing variables to $(\xi - \tau)$

$$\xi = \sigma^{\frac{1}{2}}\frac{(x - c_s t)}{R} \qquad \tau = \sigma^{\frac{3}{2}}\frac{c_s t}{R} \tag{4.13}$$

As R is a length parameter.

By using eq. (4.13) and changing the variables, we will get the following two equations in order of σ and σ^2:

$$\sigma\left\{-\frac{90\epsilon_0}{148\pi^2T^4}\frac{\partial\epsilon_1}{\partial\xi} + \frac{\partial v_1}{\partial\xi}\right\} + \sigma^2\left\{\frac{90\epsilon_0}{148\pi^2T^4}\left(-\frac{\partial\epsilon_2}{\partial\xi} + \frac{\partial\epsilon_1}{\partial\tau} + v_1\frac{\partial\epsilon_1}{\partial\xi}\right) + \frac{\partial v_2}{\partial\xi} - {c_s}^2 v_1\frac{\partial v_1}{\partial\xi}\right\} = 0 \tag{4.14}$$

$$\sigma\left\{-\frac{148\pi^2T^4c_s}{30}\frac{\partial v_1}{\partial\xi} + \frac{\epsilon_0}{c_s}\frac{\partial\epsilon_1}{\partial\xi}\right\} + \sigma^2\left\{\frac{148\pi^2T^4c_s}{30}\left(-\frac{\partial v_2}{\partial\xi} + \frac{\partial v_1}{\partial\tau} + v_1\frac{\partial v_1}{\partial\xi}\right) + \frac{\epsilon_0}{c_s}\frac{\partial\epsilon_2}{\partial\xi} - \epsilon_0 c_s v_1\frac{\partial\epsilon_1}{\partial\xi}\right\} = 0 \tag{4.15}$$

For above equations, each bracket must be equal to zero, so from (4.14) we have:

$$\sigma\left(-\frac{90\epsilon_0}{148\pi^2T^4}\frac{\partial\epsilon_1}{\partial\xi} + \frac{\partial v_1}{\partial\xi}\right) = 0 \qquad \frac{\partial v_1}{\partial\xi} = \frac{90\epsilon_0}{148\pi^2T^4}\frac{\partial\epsilon_1}{\partial\xi}$$

So, we get:

$$v_1 = \frac{90\epsilon_0}{148\pi^2T^4}\epsilon_1 \tag{4.16}$$

Replacing $(\xi - \tau)$ with $(x\text{-}t)$, then we have:

$$\frac{\partial\hat{\epsilon_1}}{\partial t} + c_s\frac{\partial\hat{\epsilon_1}}{\partial x} + \left(\frac{90\epsilon_0}{148\pi^2T^4}\right)\frac{2}{3}c_s\hat{\epsilon_1}\frac{\partial\hat{\epsilon_1}}{\partial x} = 0 \tag{4.17}$$

As $\hat{\epsilon_1} = \sigma\epsilon_1$ represents a small perturbation in ϵ

Eq.(4.17) acts as an equation for a breaking wave at finite value of temperature. Where T represents the temperature of background $(T = T_0)$, which is related to the energy density as:

$$\epsilon = \frac{37}{30}\pi^2(T^4 + {T_B}^4) \tag{4.18}$$

Using $\epsilon_0 = \frac{37}{30}\pi^2({T_0}^4 + {T_B}^4)$, and substituting in eq. (4.17), we have:

$$\frac{\partial \hat{\epsilon}_1}{\partial t} + c_s \frac{\partial \hat{\epsilon}_1}{\partial x} + \left[\frac{90 \frac{37}{30} \pi^2 ({T_0}^4 + {T_B}^4)}{148 \pi^2 T^4} \right] \frac{2}{3} c_s \hat{\epsilon}_1 \frac{\partial \hat{\epsilon}_1}{\partial x} = 0$$

$$\frac{\partial \hat{\epsilon}_1}{\partial t} + c_s \frac{\partial \hat{\epsilon}_1}{\partial x} + \left[1 + \frac{{T_B}^4}{{T_0}^4} \right] \frac{3}{4} \frac{2}{3} c_s \hat{\epsilon}_1 \frac{\partial \hat{\epsilon}_1}{\partial x} = 0$$

Finally, we have:

$$\boxed{\frac{\partial \hat{\epsilon}_1}{\partial t} + c_s \frac{\partial \hat{\epsilon}_1}{\partial x} + \left[1 + \left(\frac{T_B}{T_0} \right)^4 \right] \frac{c_s}{2} \hat{\epsilon}_1 \frac{\partial \hat{\epsilon}_1}{\partial x} = 0} \tag{4.19}$$

5 Tsallis Statistics

5.1 Energy Density and Pressure

Depending on the bosonic and fermionic single-particle distributions, we can get the energy density and the pressure of a gas of a massless bosons or fermions [4, 15]. The energy density and pressure are given as:

$$\epsilon_i = \frac{g}{(2\pi)^3} \int E_p n_i d^3p \tag{5.1}$$

$$P_i = \frac{g}{3(2\pi)^3} \int \frac{p^2}{E_p} n_i d^3p \tag{5.2}$$

Where "i" is replaced by "b" in case of bosons and "f" in case of fermions.

5.1.1 For Massless Bosons ($m = 0$)

The single particle distribution of a boson is given by:

$$n_b = \frac{1}{\left[1 + (q-1)\frac{E_p - \mu}{T}\right]^{\frac{q}{q-1}} - 1} \tag{5.3}$$

As we suppose that we have a massless system, so the mass $m = 0$ and the chemical potential $\mu = 0$. Hence, eq. (5.3) will be developed to be:

$$n_b = \frac{1}{\left[1 + (q-1)\frac{E_p}{T}\right]^{\frac{q}{q-1}} - 1} \tag{5.4}$$

As $E_p = \sqrt{p^2 + m^2}$, and $m = 0$, so $E_p = p$. Then, from eq. (5.4) we have:

$$n_b = \frac{1}{\left[1 + (q-1)\frac{p}{T}\right]^{\frac{q}{q-1}} - 1} \tag{5.5}$$

Returning to eq. (5.1), the energy density distribution for bosons is given by:

$$\epsilon_b = \frac{g}{(2\pi)^3} \int E_p \frac{1}{\left[1 + (q-1)\frac{p}{T}\right]^{\frac{q}{q-1}} - 1} d^3p \tag{5.6}$$

Similarly, from eq. (5.2), we have:

$$P_b = \frac{g}{3(2\pi)^3} \int \frac{p^2}{E_p} \frac{1}{\left[1 + (q-1)\frac{p}{T}\right]^{\frac{q}{q-1}} - 1} d^3p \tag{5.7}$$

Finally, we can express ϵ_b and P_b as following:

$$\boxed{\epsilon_b = \frac{gT^4}{2\pi^2(q-1)^3 q}\left[3\psi^{(0)}\left(\frac{3}{q}-2\right)+\psi^{(0)}\left(\frac{1}{q}\right)-3\psi^{(0)}\left(\frac{2}{q}-1\right)-\psi^{(0)}\left(\frac{4}{q}-3\right)\right]} \tag{5.8}$$

As $P_b = \frac{1}{3}\epsilon_b$, so we have:

$$\boxed{P_b = \frac{gT^4}{6\pi^2(q-1)^3 q}\left[3\psi^{(0)}\left(\frac{3}{q}-2\right)+\psi^{(0)}\left(\frac{1}{q}\right)-3\psi^{(0)}\left(\frac{2}{q}-1\right)-\psi^{(0)}\left(\frac{4}{q}-3\right)\right]} \tag{5.9}$$

5.1.2 For Massless Fermions $(m = 0)$

For non-extensive fermions, we consider the single-particle distribution function for a fermion:

$$n_f = \frac{1}{\left[1+(q-1)\frac{p}{T}\right]^{\frac{q}{q-1}}+1} \tag{5.10}$$

Similarly, as we did in equations (5.5) and (5.6), we will have:

$$\epsilon_f = \frac{g}{(2\pi)^3}\int E_p \frac{1}{\left[1+(q-1)\frac{p}{T}\right]^{\frac{q}{q-1}}+1} d^3p \tag{5.11}$$

$$P_f = \frac{g}{3(2\pi)^3}\int \frac{p^2}{E_p} \frac{1}{\left[1+(q-1)\frac{p}{T}\right]^{\frac{q}{q-1}}+1} d^3p \tag{5.12}$$

Then, for a non-extensive massless gas of fermions we have the energy density is given by:

$$\boxed{\epsilon_f = \frac{gT^4}{2\pi^2(q-1)^3 q}\left[3\phi\left(-1,1,\frac{2}{q}-1\right)-3\phi\left(-1,1,\frac{3}{q}-2\right)+\phi\left(-1,1,\frac{4}{q}-3\right)-\phi\left(-1,1,\frac{1}{q}\right)\right]} \tag{5.13}$$

and the pressure is given by:

$$\boxed{P_f = \frac{gT^4}{6\pi^2(q-1)^3 q}\left[3\phi\left(-1,1,\frac{2}{q}-1\right)-3\phi\left(-1,1,\frac{3}{q}-2\right)+\phi\left(-1,1,\frac{4}{q}-3\right)-\phi\left(-1,1,\frac{1}{q}\right)\right]} \tag{5.14}$$

5.1.3 For Massive Fermions $(m \neq 0)$

For non-extensive massive fermions, we consider the single-particle distribution function as:

$$n_f = \frac{1}{\left[1+(q-1)\frac{E_p}{T}\right]^{\frac{q}{q-1}}+1} \tag{5.15}$$

$$\epsilon_f = \frac{g}{(2\pi)^3}\int E_p \frac{1}{\left[1+(q-1)\frac{E_p}{T}\right]^{\frac{q}{q-1}}+1} d^3p \tag{5.16}$$

$$P_f = \frac{g}{3(2\pi)^3}\int \frac{p^2}{E_p} \frac{1}{\left[1+(q-1)\frac{E_p}{T}\right]^{\frac{q}{q-1}}+1} d^3p \tag{5.17}$$

Hence, for a non-extensive massive gas of fermions we have the energy density and pressure distributions are given by:

$$\epsilon_f = \frac{gT^4}{2\pi^2(q-1)^3q}\left[3\phi\left(-1,1,\frac{2}{q}-1\right)-3\phi\left(-1,1,\frac{3}{q}-2\right)+\phi\left(-1,1,\frac{4}{q}-3\right)-\phi\left(-1,1,\frac{1}{q}\right)\right]$$
$$-\frac{gm^2T^2}{4\pi^2(q-1)q}\left[\phi\left(-1,1,\frac{2}{q}-1\right)-\phi\left(-1,1,\frac{1}{q}\right)\right] \quad (5.18)$$

$$P_f = \frac{gT^4}{6\pi^2(q-1)^3q}\left[3\phi\left(-1,1,\frac{2}{q}-1\right)-3\phi\left(-1,1,\frac{3}{q}-2\right)+\phi\left(-1,1,\frac{4}{q}-3\right)-\phi\left(-1,1,\frac{1}{q}\right)\right]$$
$$-\frac{gm^2T^2}{4\pi^2(q-1)q}\left[\phi\left(-1,1,\frac{2}{q}-1\right)-\phi\left(-1,1,\frac{1}{q}\right)\right] \quad (5.19)$$

Energy density and pressure for bosons and fermions can help us to build a non-extensive MIT bag model [4], to be used in the next section.

5.2 Derivation of the Breaking Wave Equation Using Tsallis Statistics

By considering the Euler's relativistic (2.3) and the entropy conservation equations, we have

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{(v^2-1)}{(\epsilon+P)}\left(\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial t}\right)$$

Where pressure P and energy density ϵ are functions of space x and time t

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}+\mathcal{E}_1(1-v^2)\left(\frac{\partial \epsilon_{\text{bag}}}{\partial x}+v\frac{\partial \epsilon_{\text{bag}}}{\partial t}\right)=0 \quad (5.20)$$

$$\mathcal{C}_1\left(1-v^2\right)\left(v\frac{\partial \epsilon_{\text{bag}}}{\partial x}+\frac{\partial \epsilon_{\text{bag}}}{\partial t}\right)+C_2\left(\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial t}\right)=0 \quad (5.21)$$

By considering the expressions of energy density and pressure to be used in the non-extensive bag model:

$$\epsilon_{b,1} = \frac{g}{2\pi^2(q-1)^3q}\left[3\psi^{(0)}\left(\frac{3}{q}-2\right)+\psi^{(0)}\left(\frac{1}{q}\right)-3\psi^{(0)}\left(\frac{2}{q}-1\right)-\psi^{(0)}\left(\frac{4}{q}-3\right)\right]$$

$$\epsilon_{f,1} = \frac{g}{2\pi^2(q-1)^3q}\left[3\phi\left(-1,1,\frac{2}{q}-1\right)-3\phi\left(-1,1,\frac{3}{q}-2\right)+\phi\left(-1,1,\frac{4}{q}-3\right)-\phi\left(-1,1,\frac{1}{q}\right)\right]$$

$$\epsilon_{f,2} = -\frac{g}{4\pi^2(q-1)q}\left[\phi\left(-1,1,\frac{2}{q}-3\right)-\phi\left(-1,1,\frac{1}{q}\right)\right]$$

$$P_b,1=\frac{\epsilon_{b,1}}{3} \qquad P_f,1=\frac{\epsilon_{f,1}}{3} \qquad P_f,1=\epsilon_{f,2} \quad (5.22)$$

$$\epsilon_{\text{bag},1}=\epsilon_{b,1}+2\epsilon_{f,1} \qquad P_{\text{bag},1}=P_b,1+2P_f,1 \qquad \epsilon_{\text{bag},2}=2\epsilon_{f,2} \qquad P_{\text{bag},2}=2P_f,2 \quad (5.23)$$

From non-extensive bag model, we define the following terms:

$$\begin{aligned}\epsilon_{\text{bag},1} &= \epsilon_{b,1}+2\epsilon_{f,1}\\ P_{\text{bag},1} &= P_{b,1}+2P_{f,1}\\ \epsilon_{\text{bag},2} &= 2\epsilon_{f,2}\\ P_{\text{bag},2} &= 2P_{f,2}\end{aligned} \quad (5.24)$$

$$\mathcal{C}_1 = m^2\epsilon_{\text{bag},2} + 2T^2\epsilon_{\text{bag},1} \tag{5.25}$$

$$\mathcal{C}_2 = \left(m^2\epsilon_{\text{bag},2} + \frac{2}{3}\epsilon_{\text{bag},1}T^2\right)\left(4T^2\epsilon_{\text{bag},1} + 2m^2T^2\epsilon_{\text{bag},2}\right) \tag{5.26}$$

$$\mathcal{E}_1 = \frac{\left(m^2\epsilon_{\text{bag},2} + \frac{2}{3}\epsilon_{\text{bag},1}T^2\right)}{\left(m^2\epsilon_{\text{bag},2} + 2T^2\epsilon_{\text{bag},1}\right)\left(\frac{4}{3}T^4\epsilon_{\text{bag},1} + 2m^2T^2\epsilon_{\text{bag},2}\right)} \tag{5.27}$$

We now reformulate equations (5.20) and (5.21) using the following dimensionless variables:

$$\hat{\epsilon} = \frac{\epsilon}{\epsilon_0} \qquad \hat{v} = \frac{v}{c_s}$$

Where ϵ_0 is the reference energy density and then by expanding the previous equations in power of σ:

$$\hat{\epsilon} = 1 + \sigma\epsilon_1 + \sigma^2\epsilon_2 + \mathcal{O}(\sigma^3) \qquad \hat{v} = 1 + \sigma v_1 + \sigma^2 v_2 + \mathcal{O}(\sigma^3)$$

By changing the variables from $(x,\, t)$ to $(\xi,\, \tau)$ following that:

$$\xi = \sigma^{\frac{1}{2}}\frac{(x - c_s t)}{R} \qquad \tau = \sigma^{\frac{3}{2}}\frac{c_s t}{R} \tag{5.28}$$

R is a length parameter.

From the first order of σ, we will have:

$$c_s^2\frac{\partial v_1}{\partial \xi} = \epsilon_0\mathcal{E}_1\frac{\partial \epsilon_1}{\partial \xi} \qquad \mathcal{C}_2\frac{\partial v_1}{\partial \xi} = \epsilon_0\mathcal{C}_1\frac{\partial \epsilon_1}{\partial \xi} \tag{5.29}$$

Now we reformulate and solve the last two equations to get the velocity of sound:

$$c_s^2 = \frac{\mathcal{C}_2}{\mathcal{C}_1}\mathcal{E}_1 \tag{5.30}$$

Depending on the second order of σ

$$\frac{\partial \epsilon_1}{\partial \tau} + \frac{2\epsilon_0\epsilon_1\epsilon_{\text{bag},1}}{3m^4\epsilon^2_{\text{bag},2} + 8m^2T^2\epsilon_{\text{bag},1}\epsilon_{\text{bag},2} + 4T^4\epsilon^2_{\text{bag},1}}\frac{\partial \epsilon_1}{\partial \xi} = 0 \tag{5.31}$$

The previous equation provides the expectation to the energy density of the first order perturbation.

By replacing $(\xi - \tau)$ with $(x - t)$ we obtain the following:

$$\boxed{\frac{\partial \hat{\epsilon}_1}{\partial t} + c_s\frac{\partial \hat{\epsilon}_1}{\partial x} + \frac{2c_s\epsilon_0\epsilon_{\text{bag},1}\hat{\epsilon}_1}{3m^4\epsilon^2_{\text{bag},2} + 8m^2T^2\epsilon_{\text{bag},1}\epsilon_{\text{bag},2} + 4T^4\epsilon^2_{\text{bag},1}}\frac{\partial \hat{\epsilon}_1}{\partial x} = 0} \tag{5.32}$$

At constant temperature, we will have the coefficient of nonlinear term as:

$$B_c = \frac{2\epsilon_0\epsilon_{\text{bag},1}}{3m^4\epsilon^2_{\text{bag},2} + 8m^2T^2\epsilon_{\text{bag},1}\epsilon_{\text{bag},2} + 4T^4\epsilon^2_{\text{bag},1}} \tag{5.33}$$

So, eq. (5.32) will become:

$$\boxed{\frac{\partial \epsilon_1}{\partial \tau} + B_c\epsilon_1\frac{\partial \epsilon_1}{\partial \xi} = 0} \tag{5.34}$$

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