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FINAL REPORT ON THE INTEREST PROGRAMME

Tsallis thermodynamic variables and their applications in high-energy collision physics

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Abstract

Conventional Boltzmann-Gibbs statistical mechanics has contributed a lot to the understanding of many natural phenomena. Non-extensive or Tsallis statistical mechanics has an even broader range of applicability in social and natural sciences, particularly, in Physics. The aim of this report is to introduce some of the several applications of the framework of Tsallis statistical mechanics, highlight the basic concepts involved in this framework, derive the thermodynamic variables for massless classical and quantum particles, and show the solution to the ideal hydrodynamic equation as well as the numerical computation for the pressure of massive particles in the framework of both Tsallis and Boltzmann-Gibbs. Also, some special functions will be mentioned throughout the derivations along with some illustrative figures generated using MATHEMATICA.
Keywords: Tsallis statistics, Entropic index, Thermodynamic variables

Introduction & Literature Review

In 1988, Constantino Tsallis proposed a generalization for the well-known Boltzmann-Gibbs (BG) statistical mechanics based on the concepts of multifractals [1]. This generalized statistical mechanics is called Tsallis statistical mechanics. It is mainly based on the functional form of the entropy in terms of probabilities S_q where q is called the entropic index, and for $q = 1$, the usual entropy S_{BG} is retrieved. This generalized entropy S_q -unlike S_{BG}- can have the property of being *nonadditive* meaning that the entropy of a combined system does not equal the sum of the entropies of the constituent independent subsystems. Nowadays, several systems in nature have been studied which happened to follow the framework of Tsallis statistics, not only in Physics, but also in other various fields such as Economics, Biology, and Chemistry. Among those various applications of this generalized entropy S_q or Tsallis statistics, the resultant single particle distribution functions -as shall be later specified- are *derivable* power-laws that recover the exponential ones of BG in the $q = 1$ limit. Those power-like distributions are relevant in several physical contexts mainly the ones where long-range interactions are unavoidable [2]. Some of those endless applications of Tsallis statistics in some of the many disciplines of Physics, in particular High Energy Physics, are as follows.

In High-Energy Physics, several High-Energy collisions such as proton-proton collision, and electron-positron annihilation give transverse momenta spectrum that can be perfectly reproduced using Tsallis statistics for $1.2 > q > 1$ rather than Boltzmann statistics [3, 4]. Additionally, in lead-lead collision, quark-gluon plasma is created at huge energy density, and the thermodynamic variables such as entropy density and pressure can be determined at kinetic freeze out from the transverse momentum spectrum. This spectrum -as previously mentioned- typically follows the Tsallis distribution with $q \approx 1.139$ at centre of mass energy $\sqrt{s} = 2.76$ TeV and with $q \approx 1.145$ at $\sqrt{s} = 5.02$ TeV. Moreover, with the transverse momentum spectrum, one can obtain the thermodynamic variables which help to find out the life time of different collision phases such as the quark-gluon plasma stage [5]. Besides, in diffusion process of Charm Quarks (CQs), they experience collisional and radiative interactions in a background of quark-gluon plasma. Although the background equilibrium distribution obeys the Boltzmann statistics, the CQs equilibrium distribution is well described by Tsallis statistics with $q \approx 1.1$ [6, 7].

Moving to other disciplines of Physics starting with Astrophysics, it was shown that the velocity distributions of galaxies clusters are perfectly fitted with Tsallis like distributions [8, 9]. Moreover, the temperature distribution of cosmic background radiation is well described by a q-Gaussian, i.e. following Tsallis statistics [10]. In addition, in Condensed Matter Physics, some of the complicated problems such as Manganites and ferro-paramagnetic phase transitions have been tackled using phenomenological Tsallis distributions with q as a fitting parameter

[11–13]. Additionally, in Geophysics, the q-exponential distributions, are obeyed in several contexts including the distances between successive earthquakes with $q < 1$ and aftershocks rate with $q > 1$ [14, 15].

To reemphasize, these examples constitute a small subset of wide range of applications of Tsallis statistics, and further ones can be found in this bibliography [16].

In this report, we aim to derive the Tsallis thermodynamic variables for different types of particles (Quantum & Classical). The relationship between the thermodynamic variables (e.g that between pressure and energy density and/or entropy density and energy density) is called the equation of state. Hence, we will also demonstrate how these thermodynamic variables are used in the equation of state that is extensively used to study the evolution of quark-gluon medium formed after High Energy collisions.
The report is structured as follows. We will first introduce the basic conceptual and mathematical formulation of Tsallis statistics in *section* 1. Then, the detailed derivation of some of the Tsallis thermodynamic variables in massless case at zero chemical potential will be covered in *section* 2. The solution of the ideal hydrodynamic equation in massless case will be presented along with discussion of the obtained results with some illustrations in *section* 3. Eventually, we will conclude and summarize in *section* 4.

1 Mathematical & Conceptual Framework

In this section, we will cover the basic formulation of Tsallis statistics as well as introducing the the basic definitions and concepts needed throughout our discussion. Also note that natural units ($k_B = \hbar = c = 1$) will be used throughout.

1.1 Tsallis Entropy & The Entropic Index (q)

We start with the foundation of the whole theory, which is the Tsallis generalized entropy defined as:

$$S_q = -\sum_i p_i^q \, \frac{p_i^{1-q} - 1}{1 - q} \tag{1}$$

where p_i is the usual microstates probability, and the sum runs over all possible energy eigenstates of our system (Assuming discrete spectrum).
The new parameter appearing here q or the entropic index is of main significance not only because it allows us to recover the ordinary Boltzmann-Gibbs entropy in the limit $q \to 1$, but also due to its physical significance since it can be related to other physical quantities depending on the context. One interpretation of q is that it is related to fluctuations in the temperature -around a mean temperature T_0- of microscopic subsystems which together construct a macroscopic system that is in contact with a reservoir at that mean temperature T_0 [17]. Entropic index can be also related to the heat capacity C of a finite reservoir ($q = 1 - \frac{1}{C}$) which in the Boltzmann limit of infinite reservoir, i.e infinite heat capacity, we see that $q \to 1$ as indicated before [18]. These interpretations mainly indicate that the entropic index is not an ad-hoc or a fitting parameter, but an essential parameter that indicates an essential physical property -such as temperature fluctuations- of the system under consideration.

1.2 Tsallis Probability

Now, since we have a formula for the entropy in-terms of probabilities, we can proceed to derive the equilibrium probability distribution for a grand canonical ensemble which results from grand potential extremization using the method of Lagrange multiplier.

The details of the procedures are as follows. **Firstly**, *we express the entropy in terms of microstates probabilities*, which in our case is given by (1). **Secondly**, *We define the grand potential Ω:*

$$\Omega = \langle H\rangle - TS - \mu\langle N\rangle$$

and demand that it has an extrema at equilibrium. Where the expectation values of any quantity are given by: $\langle \mathbf{A}\rangle = \sum_{\mathbf{i}} \mathbf{p_i^q A_i}$, where A_i is the i^{th} eigenvalue of this quantity [19]. This type of statistics is also called Tsallis-2 statistics.

Finally, we extermize this potential using Lagrange's multipliers method, which means we look for an extrema of the grand potential, p_i', subject to the normalization constraints $\sum_{\mathbf{i}} \mathbf{p_i} = \mathbf{1}$. We start by defining

$$\Omega' = \Omega - \lambda\phi$$

where: 1) λ is our Lagrange multiplier, and 2) $\phi = \sum_i p_i - 1 = 0$, so we have that:

$$\Omega' = \langle H\rangle - TS - \mu\langle N\rangle - \lambda(\sum_i p_i - 1)$$

$$\Omega' = \sum_i p_i^q[E_i + \frac{T}{1-q}(p_i^{1-q} - 1) - \mu N_i - \lambda p_i^{1-q}] - \lambda$$

and *eventually* we demand

$$\frac{\partial\Omega'}{\partial p_j}|_{p_j=p_j'} = 0$$

Which gives:

$$\sum_i q\, p_i^{q-1}\, \delta_{ij}[E_i + \frac{T}{1-q}(p_i^{1-q} - 1) - \mu N_i - \lambda p_i^{1-q}]|_{p_j=p_j'} + \sum_i p_i^q[T - \lambda(1-q)]p_i^{-q}\delta_{ij}|_{p_j=p_j'} = 0$$

$$q\, p_j'^{q-1}\, [E_j + \frac{T}{1-q}(p_j'^{1-q} - 1) - \mu N_j - \lambda p_j'^{1-q}] + p_j'^q[T - \lambda(1-q)]p_j'^{-q} = 0$$

This can be further simplified as follows:

$$q(E_j - \frac{T}{1-q} - \mu N_j)p_j'^{-(1-q)} = \lambda - \frac{T}{1-q}$$

$$\frac{T}{1-q}(\frac{1-q}{T}E_j - 1 - \frac{1-q}{T}\mu N_j)p_j'^{-(1-q)} = -\frac{T}{q(1-q)}(1-(1-q)\frac{\lambda}{T})$$

$$(1-(1-q)\frac{E_j - \mu N_j}{T})p_j'^{-(1-q)} = (1-(1-q)\frac{\lambda}{T})/q$$

We can further define that $(1-(1-q)\frac{\lambda}{T})/q = Z^{1-q}$ which gives:

$$Z^{1-q}p_j'^{1-q} = 1-(1-q)\frac{E_j - \mu N_j}{T}$$

$$p_j' = \frac{1}{Z}\Big[1-(1-q)\frac{E_j - \mu N_j}{T}\Big]^{\frac{1}{1-q}} \Rightarrow \boxed{p_j' \propto \Big[1-(1-q)\frac{E_j - \mu N_j}{T}\Big]^{\frac{1}{1-q}}}$$

1.3 Tsallis & Boltzmann single particle distributions

Since we have the probabilities, we can now derive the single particle distribution functions for classical and quantum cases using the standard methods of Classical Statistical Mechanics. We limit our discussions to the zeroth order distributions here and move on to their use in next section [20–22]:

$$f_{\text{MB}} = \left[1 + (q-1)\frac{E_p - \mu}{T}\right]^{\frac{-1}{q-1}} \quad \text{(Maxwell-Boltzmann (MB), Tsallis)} \tag{2}$$

$$n_{\text{MB}} = e^{-\frac{E_p-\mu}{T}} \quad \text{(Maxwell-Boltzmann, Boltzmann-Gibbs)} \tag{3}$$

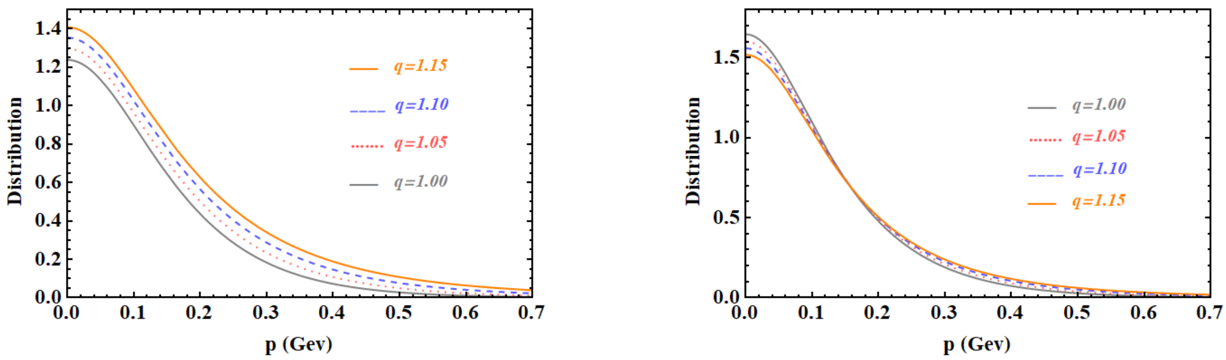
$$f_{\text{FD}} = \frac{1}{\left[1 + (q-1)\frac{E_p-\mu}{T}\right]^{\frac{q}{q-1}} + 1} \quad \text{(Fermi-Dirac (FD), Tsallis)} \tag{4}$$

$$n_{\text{FD}} = \frac{1}{e^{\frac{E_p-\mu}{T}} + 1} \quad \text{(Fermi-Dirac, Boltzmann-Gibbs)} \tag{5}$$

$$f_{\text{BE}} = \frac{1}{\left[1 + (q-1)\frac{E_p-\mu}{T}\right]^{\frac{q}{q-1}} - 1} \quad \text{(Bose-Einstein (BE), Tsallis)} \tag{6}$$

$$n_{\text{BE}} = \frac{1}{e^{\frac{E_p-\mu}{T}} - 1} \quad \text{(Bose-Einstein, Boltzmann-Gibbs)} \tag{7}$$

where the first item in the tuple specifies the type of the particles (Particle distribution), while the second item identify the type of the used statistics (Energy distribution). Before proceeding in the calculations of the thermodynamic variables, we illustrate the differences between Boltzmann-Gibbs ($q = 1$) and Tsallis distribution functions in Figure. 1 for several q values. [1]



(a) Maxwell-Boltzmann (b) Bose-Einstein

Figure 1: MB & BE distributions for π^+ particles at T= 0.1 GeV

2 Thermodynamic Variables

The general formulas for the different thermodynamic variables can be obtained in large volume limit of the standard ones known from Classical Statistical Mechanics with the specification of Tsallis-2 expectation values mentioned before [23].

[1] The distributions are scaled by a factor of 5 to give a better looking y-axis.

These variables are generally given as follows in the case of spherical symmetry in 3D momentum space:

$$P = \frac{4\pi g}{3(2\pi)^3} \int_0^\infty \frac{p^4}{E_p} f^q \, dp \tag{8}$$

$$\epsilon = \frac{E}{V} = \frac{4\pi g}{(2\pi)^3} \int_0^\infty p^2 E_p f^q \, dp \tag{9}$$

$$s = \frac{S}{V} = -\frac{4\pi g}{(2\pi)^3} \int_0^\infty p^2 [f^q \ln_q f - f] \, dp \tag{10}$$

$$n = \frac{N}{V} = \frac{4\pi g}{(2\pi)^3} \int_0^\infty p^2 f^q \, dp \tag{11}$$

where g is the degeneracy factor, P is the pressure, ϵ is the energy density, s is the entropy density, n is the number density, and the q-logarithm $\ln_q f$ is defined as:

$$\ln_q f = \frac{1 - f^{1-q}}{q-1}$$

with f being one of the single particles distribution functions defined in (2-7), but recall that for Boltzmann-Gibbs statistics we have ($q = 1$). Also, $E_p = \sqrt{m^2 + p^2}$ is the single particle energy, μ is the chemical potential, and we may sometimes use the identification $\delta q = q - 1$ for simplicity.

One last thing to notice before going into the details of the calculation is that those variables should be finite to have any physical significance. This fact implies an upper bound for the entropic index in Tsallis cases [2], which can be figured out from the integrand of the pressure for example at large momentum values that is approximately ($\approx \frac{p^4}{p} p^{\frac{q}{1-q}} = p^{\frac{3-2q}{1-q}}$). Now, to have convergent pressure, we want the momentum in the integrand to be of degree at most (-1), which eventually gives ($1 < q < \frac{4}{3}$).

2.1 Classical Massless Particles

Since we are considering the case of $\underline{m = 0 \ \& \ \mu = 0}$, the single particle energy E_p ends up as:

$$E_p = \sqrt{m^2 + p^2} = p$$

Taking all these points into consideration, we start with P as follows:

$$P = \frac{4\pi g}{3(2\pi)^3} \int_0^\infty \frac{p^4}{p} \Big[1 + \delta q \frac{p}{T}\Big]^{\frac{-q}{\delta q}} dp = \frac{g}{6\pi^2} \int_0^\infty p^3 \Big[1 + \delta q \frac{p}{T}\Big]^{\frac{-q}{\delta q}} dp \tag{12}$$

We define ($\delta q \frac{p}{T} = u$), so:

$$P = \frac{g T^4}{6\pi^2 \delta q^4} \int_0^\infty u^3 [1+u]^{-\frac{q}{\delta q}} \, du$$

This integral can be evaluated using the **Beta function**, defined as:

$$\mathbf{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty u^{x-1} [1+u]^{-(x+y)} \, du \tag{13}$$

In our case: $\underline{x = 4}$ & $y + x = \frac{q}{\delta q} \rightarrow \underline{y = \frac{q}{\delta q} - 4}$, which gives:

$$P = \frac{g T^4}{6\pi^2 \delta q^4} \frac{\Gamma(4)\Gamma(\frac{q}{\delta q} - 4)}{\Gamma(\frac{q}{\delta q})} = \frac{g T^4}{6\pi^2 \delta q^4} \frac{3! \, \Gamma(\frac{q}{\delta q} - 4)}{(\frac{q}{\delta q} - 1)(\frac{q}{\delta q} - 2)(\frac{q}{\delta q} - 3)(\frac{q}{\delta q} - 4)\Gamma(\frac{q}{\delta q} - 4)}$$
$$= \frac{g T^4}{\pi^2} \frac{1}{(q - \delta q)(q - 2\delta q)(q - 3\delta q)(q - 4\delta q)}$$

[2]The lower bound as far as we are concerned is 1.

Where we used the property of the Gamma function ($\Gamma(x+1) = x\Gamma(x)$), but we have that: $q - \delta q = q - q + 1 = 1$, similarly we reach: $q - m\delta q = 1 - (m-1)\delta q$, where $m = 2, 3, 4$; hence:

$$P = \frac{gT^4}{\pi^2(1-\delta q)(1-2\delta q)(1-3\delta q)} = \boxed{\frac{gT^4}{6\pi^2(1-\delta q)(\frac{1}{2}-\delta q)(\frac{1}{3}-\delta q)}} \tag{14}$$

Moving on to **the second thermodynamical variable**, Energy density:

$$\epsilon = \frac{g}{2\pi^2}\int_0^\infty p^3\Big[1 + \delta q \frac{p}{T}\Big]^{\frac{-q}{\delta q}}\, dp$$

That is exactly the case we had in line 12, the only difference is that $\frac{1}{6} \to \frac{1}{2}$; hence, we end up with:

$$\epsilon = 3P = \boxed{\frac{gT^4}{2\pi^2(1-\delta q)(\frac{1}{2}-\delta q)(\frac{1}{3}-\delta q)}} \tag{15}$$

Now for **the third thermodynamical variable**, Entropy density:

$$s = -\frac{g}{2\pi^2}\int_0^\infty p^2[\frac{f^q - f}{\delta q} - f]\, dp = \frac{g}{2\pi^2}\int_0^\infty p^2\frac{qf - f^q}{\delta q}\, dp = \frac{g}{2\pi^2\delta q}(qI_1 - I_2) \tag{16}$$

Where we solve each integral separately as follows:

$$I_1 = \int_0^\infty p^2 f\, dp = \int_0^\infty p^2\Big[1 + \delta q\frac{p}{T}\Big]^{\frac{-1}{\delta q}}\, dp$$

Again, we define ($\delta q\frac{p}{T} = u$), which gives:

$$I_1 = \frac{T^3}{\delta q^3}\int_0^\infty u^2[1+u]^{-\frac{1}{\delta q}}\, du$$

That is again the Beta function we defined in equation 13, but in this case: $\underline{x = 3}$ & $y + x = \frac{1}{\delta q} \to \underline{y = \frac{1}{\delta q} - 3}$, which gives:

$$I_1 = \frac{T^3}{\delta q^3}\frac{\Gamma(3)\Gamma(\frac{1}{\delta q}-3)}{\Gamma(\frac{1}{\delta q})} = \frac{2T^3\Gamma(\frac{1}{\delta q}-3)}{\delta q^3(\frac{1}{\delta q}-1)(\frac{1}{\delta q}-2)(\frac{1}{\delta q}-3)\Gamma(\frac{1}{\delta q}-3)} = \underline{\frac{T^3}{3(1-\delta q)(\frac{1}{2}-\delta q)(\frac{1}{3}-\delta q)}}$$

For the second integral:

$$I_2 = \int_0^\infty p^2 f^q = \int_0^\infty p^2\Big[1 + \delta q\frac{p}{T}\Big]^{\frac{-q}{\delta q}}\, dp = \frac{T^3}{\delta q^3}\int_0^\infty u^2[1+u]^{\frac{-q}{-\delta q}}\, du \tag{17}$$

That integral is again the Beta function with: $\underline{x = 3}$ & $x + y = \frac{q}{\delta q} \to \underline{y = \frac{q}{\delta q} - 3}$, so we get:

$$I_2 = \frac{T^3}{\delta q^3}\frac{\Gamma(3)\Gamma(\frac{q}{\delta q}-3)}{\Gamma(\frac{q}{\delta q})} = \frac{2T^3\Gamma(\frac{q}{\delta q}-3)}{(q-\delta q)(q-2\delta q)(q-3\delta q)\Gamma(\frac{q}{\delta q}-3)} = \underline{\frac{T^3}{(1-\delta q)(\frac{1}{2}-\delta q)}} \tag{18}$$

We substitute back into equation 16:

$$s = \frac{g}{2\pi^2\delta q}\frac{T^3}{(1-\delta q)(\frac{1}{2}-\delta q)}(\frac{q}{(1-3\delta q)} - 1) = \frac{gT^3}{2\pi^2(1-\delta q)(\frac{1}{2}-\delta q)}\frac{4\delta q}{3\delta q(\frac{1}{3}-\delta q)}$$

$$s = \boxed{\frac{2gT^3}{3\pi^2(1-\delta q)(\frac{1}{2}-\delta q)(\frac{1}{3}-\delta q)}} \tag{19}$$

Finally, we carry out the same procedures for **the fourth thermodynamical variable** Number density:

$$n = \frac{g}{2\pi^2}\int_0^\infty p^2\Big[1 + \delta q\frac{p}{T}\Big]^{\frac{-q}{\delta q}}\, dp$$

This is exactly the value of I_2 which we evaluated in 17 & 18, up to a factor of ($\frac{g}{2\pi^2}$), so we end up with:

$$n = \frac{g}{2\pi^2}I_2 = \boxed{\frac{gT^3}{2\pi^2(1-\delta q)(\frac{1}{2}-\delta q)}} \tag{20}$$

2.2 Quantum Massless Particles

Due to the limited space, we will focus on the pressure (8) of **massless** quantum particles that follow Tsallis statistics at **Zero chemical potential**. Other thermodynamic variables can be similarly calculated.

2.2.1 Fermions

We start with **Fermions** for which the single particle distribution function is given by (4).
Again, since we are considering the case of $m = 0$ & $\mu = 0$, then the single particle energy $E_p = p$ as before.
We move on to calculate the pressure for Fermions as follows.

$$P = \frac{g}{6\pi^2}\int_0^\infty p^3 n_f \, dp = \frac{g}{6\pi^2}\int_0^\infty \frac{p^3}{\left[1+(q-1)\frac{p}{T}\right]^{\frac{q}{q-1}}+1}\, dp$$

We first note that the integrand can be further simplified using the fact that:

$$\frac{1}{\left[1+(q-1)\frac{p}{T}\right]^{\frac{q}{q-1}}+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{\left[1+(q-1)\frac{p}{T}\right]^{\frac{(n+1)q}{q-1}}}$$

So we can switch the sum with the integral to obtain:

$$P = \frac{g}{6\pi^2}\sum_{n=0}^{\infty}(-1)^n\int_0^\infty p^3\left[1+(q-1)\frac{p}{T}\right]^{\frac{-(n+1)q}{q-1}}\, dp \tag{21}$$

To evaluate the integral, we redo the procedures we did for the classical particles as follows. Define $\Big((q-1)\frac{p}{T} = u \;\rightarrow\; dp = \frac{T}{q-1}du\Big)$, so:

$$\begin{aligned}
P &= \frac{gT^4}{6\pi^2(q-1)^4}\sum_{n=0}^{\infty}(-1)^n\int_0^\infty u^3[1+u]^{\frac{-(n+1)q}{q-1}}\, du \\
&= \frac{gT^4}{6\pi^2(q-1)^4}\sum_{n=0}^{\infty}(-1)^n\,\mathbf{B}(4, \frac{(n+1)q}{q-1}-4) \\
&= \frac{gT^4}{6\pi^2(q-1)^4}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(4)\Gamma(\frac{(n+1)q}{q-1}-4)}{\Gamma(\frac{(n+1)q}{q-1})}
\end{aligned}$$

Using the properties of the Gamma function we simplify:

$$\begin{aligned}
P &= \frac{gT^4}{\pi^2(q-1)^4}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\frac{(n+1)q}{q-1}-4)}{(\frac{(n+1)q}{q-1}-1)(\frac{(n+1)q}{q-1}-2)(\frac{(n+1)q}{q-1}-3)(\frac{(n+1)q}{q-1}-4)\Gamma(\frac{(n+1)q}{q-1}-4)} \\
&= \frac{gT^4}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(nq+1)(nq-q+2)(nq-2q+3)(nq-3q+4)} \\
&= \frac{gT^4}{\pi^2 q^4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\frac{1}{q})(n+\frac{2}{q}-1)(n+\frac{3}{q}-2)(n+\frac{4}{q}-3)} = \frac{gT^4}{\pi^2 q^4}\sum_{n=0}^{\infty}(-1)^n\prod_{i=0}^{i=3}\frac{1}{\Big(n+\frac{i+1}{q}-i\Big)}
\end{aligned}$$

However, we can do partial fraction which simplifies the product of fractions to be:

$$\prod_{i=0}^{i=3}\frac{1}{\Big(n+\frac{i+1}{q}-i\Big)} = \frac{q^3}{6(q-1)^3}\Big(-\frac{1}{(n+\frac{1}{q})}+\frac{3}{(n+\frac{2}{q}-1)}-\frac{3}{(n+\frac{3}{q}-2)}+\frac{1}{(n+\frac{4}{q}-3)}\Big)$$

So, the Pressure becomes:

$$P = \frac{gT^4}{6\pi^2(q-1)^3q}\sum_{n=0}^{\infty}(-1)^n\Big(-\frac{1}{(n+\frac{1}{q})}+\frac{3}{(n+\frac{2}{q}-1)}-\frac{3}{(n+\frac{3}{q}-2)}+\frac{1}{(n+\frac{4}{q}-3)}\Big)$$
$$= \frac{gT^4}{6\pi^2(q-1)^3q}\Bigg(-\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\alpha_1)}+3\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\alpha_2)}-3\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\alpha_3)}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+\alpha_4)}\Bigg)$$

Where for simplicity we defined:

$$\boxed{\alpha_1=\frac{1}{q},\ \alpha_2=\frac{2}{q}-1,\ \alpha_3=\frac{3}{q}-2,\ \&\ \alpha_4=\frac{4}{q}-3} \tag{22}$$

Now, we use the **Lerch transcendent function** defined as:

$$\mathbf{\Phi}(z,s,\alpha)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s} \tag{23}$$

and in our case we have ($z=-1$ & $s=1$), while α depends on each sum, so we end up with:

$$\boxed{P_f=\frac{gT^4}{6\pi^2(q-1)^3q}\Bigg(-\mathbf{\Phi}(-1,1,\alpha_1)+3\mathbf{\Phi}(-1,1,\alpha_2)-3\mathbf{\Phi}(-1,1,\alpha_3)+\mathbf{\Phi}(-1,1,\alpha_4)\Bigg)} \tag{24}$$

2.2.2 Bosons

For the case of **Bosons**, the single particle distribution function is given from (6). With the same previous considerations of $m=0$ and $\mu=0$ we calculate the pressure as follows:

$$P=\frac{g}{6\pi^2}\int_0^{\infty}p^3 n_b\,dp=\frac{g}{6\pi^2}\int_0^{\infty}\frac{p^3}{\Big[1+(q-1)\frac{p}{T}\Big]^{\frac{q}{q-1}}-1}\,dp$$

A similar trick can be performed to simplify the integrand by using:

$$\frac{1}{\Big[1+(q-1)\frac{p}{T}\Big]^{\frac{q}{q-1}}-1}=\sum_{n=0}^{\infty}\frac{1}{\Big[1+(q-1)\frac{p}{T}\Big]^{\frac{(n+1)q}{q-1}}}$$

Then we get:

$$P=\frac{g}{6\pi^2}\sum_{n=0}^{\infty}\int_0^{\infty}p^3\Big[1+(q-1)\frac{p}{T}\Big]^{\frac{-(n+1)q}{q-1}}\,dp$$

The integral is exactly the same one we found in (21) along with the same lines of simplification after it, so we can conclude that:

$$P=\frac{gT^4}{6\pi^2(q-1)^4}\sum_{n=0}^{\infty}\mathbf{B}(4,\frac{(n+1)q}{q-1}-4)=\frac{gT^4}{6\pi^2(q-1)^4}\sum_{n=0}^{\infty}\frac{\Gamma(4)\Gamma(\frac{(n+1)q}{q-1}-4)}{\Gamma(\frac{(n+1)q}{q-1})}$$
$$=\frac{gT^4}{\pi^2q^4}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{q})(n+\frac{2}{q}-1)(n+\frac{3}{q}-2)(n+\frac{4}{q}-3)}$$
$$=\frac{gT^4}{6\pi^2(q-1)^3q}\Bigg(-\sum_{n=0}^{\infty}\frac{1}{(n+\alpha_1)}+3\sum_{n=0}^{\infty}\frac{1}{(n+\alpha_2)}-3\sum_{n=0}^{\infty}\frac{1}{(n+\alpha_3)}+\sum_{n=0}^{\infty}\frac{1}{(n+\alpha_4)}\Bigg)$$

Where the α_s are as defined in (22), but now in order to obtain a closed form -nice looking- solution, we add and subtract ($\sum_{n=0}^{\infty}\frac{1}{n+1}$) for each sum to get:[3]

$$P = \frac{gT^4}{6\pi^2(q-1)^3 q}\Bigg(\sum_{n=0}^{\infty}\Big[\frac{1}{n+1}-\frac{1}{(n+\alpha_1)}\Big] - 3\sum_{n=0}^{\infty}\Big[\frac{1}{n+1}-\frac{1}{(n+\alpha_2)}\Big] + 3\sum_{n=0}^{\infty}\Big[\frac{1}{n+1}-\frac{1}{(n+\alpha_3)}\Big] - \sum_{n=0}^{\infty}\Big[\frac{1}{n+1}-\frac{1}{(n+\alpha_4)}\Big]\Bigg)$$

Here, each sum can represented by the **Digamma function** defined as:

$$\boldsymbol{\psi}(z) = -\gamma + \sum_{n=0}^{\infty}\Big[\frac{1}{n+1}-\frac{1}{n+z}\Big] \tag{25}$$

Where γ is the Euler–Mascheroni constant, which shall cancel out after simplification (Expected).
Now the pressure becomes:

$$\boxed{P_b = \frac{gT^4}{6\pi^2(q-1)^3 q}\Big(\boldsymbol{\psi}(\alpha_1) - 3\boldsymbol{\psi}(\alpha_2) + 3\boldsymbol{\psi}(\alpha_3) - \boldsymbol{\psi}(\alpha_4)\Big)} \tag{26}$$

2.3 On Massive Particles

The calculations of the thermodynamic variables in the massive case using the Tsallis statistics can be found with the help of the contour integration technique [24]. However, this is beyond the scope of the present report, and we will employ numerical techniques to evaluate thermodynamic quantities for a gas of massive particles in the next section.

3 Results & Discussion

We begin with the ideal hydrodynamic equation mentioned in the introduction, which is given as [25]:

$$\frac{d\epsilon}{d\tau} = -\frac{\epsilon + P}{\tau} \tag{27}$$

Where τ is the proper-time. In massless cases, the relation between the pressure P and the energy density ϵ, for both the Boltzmann-Gibbs and the Tsallis case, is simply $P = \frac{1}{3}\epsilon$ as can be seen from the general definition in (8 & 9) with $E_p = p$. The solution of this separable differential equation for the initial condition $\epsilon(\tau_0)$ is as follows:

$$\frac{d\epsilon}{d\tau} = \frac{-4}{3}\frac{\epsilon}{\tau} \rightarrow \frac{d\epsilon}{\epsilon} = \frac{-4}{3}\frac{d\tau}{\tau} \rightarrow \ln\epsilon = \ln\frac{1}{\tau^{\frac{4}{3}}} + \ln c_0 \Rightarrow \underline{\epsilon = \frac{c_0}{\tau^{\frac{4}{3}}}}$$

where we integrated both sides in the second step to obtain the third, and we are free to take our constant of that form so that when the initial condition is invoked, the constant is simply:

$$\epsilon(\tau_0) = \frac{c_0}{\tau_0^{\frac{4}{3}}} \Rightarrow \underline{c_0 = \epsilon(\tau_0)\tau_0^{\frac{4}{3}}}$$

So the energy density evolve in time as indicated in Figure. 2, according to:

$$\boxed{\epsilon = \epsilon(\tau_0)\Big(\frac{\tau_0}{\tau}\Big)^{\frac{4}{3}}} \tag{28}$$

[3] This is nothing but the addition of an overall ZERO.

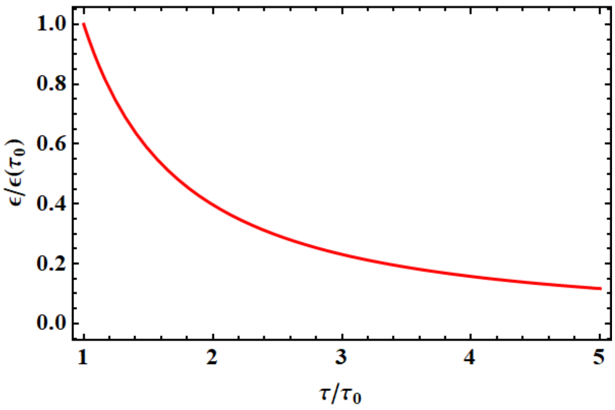
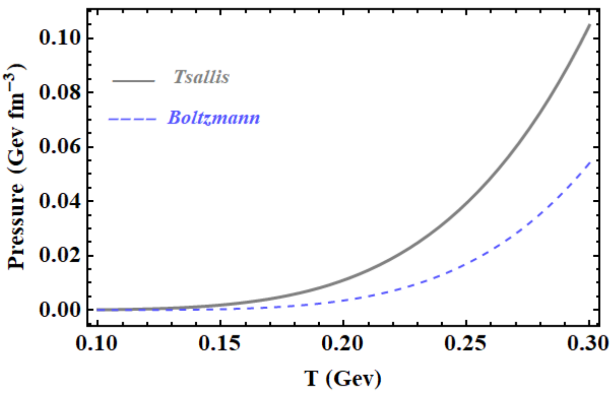
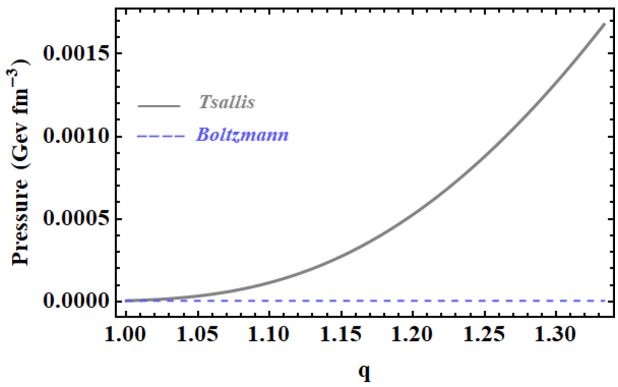


Figure 2: The time evolution of energy density for massless particles.

Now, for the massive particles, the pressure dependence on T and q separately are described in Figure. 3 for the case of protons (Fermions) in both Tsallis and Boltzmann statistics.



(a) Pressure as function of T for $q = 1.1$



(b) Pressure as function of q for $T = 0.1$ GeV

Figure 3: Pressure dependence for massive particles (Protons)

These two variations can be combined in the following 3d plot for the pressure in the case of Fermi-Dirac distribution (i.e. Protons).

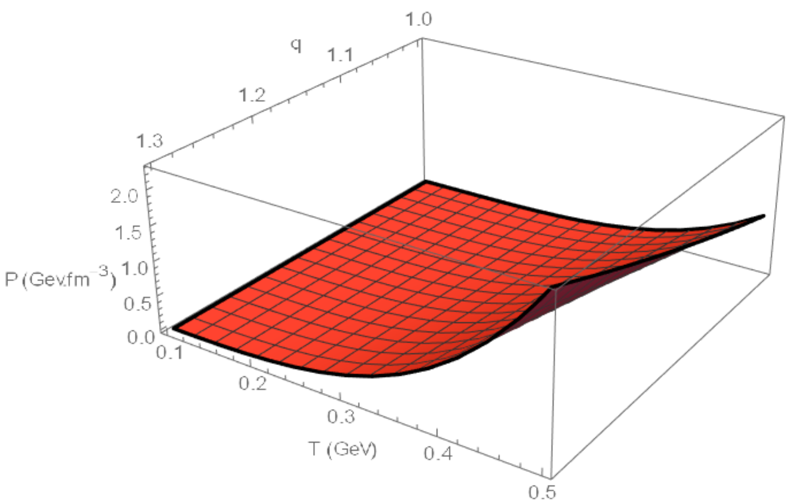


Figure 4: Pressure as function of both T and q

Now, we can see how the Tsallis statistics differ from the Boltzmann one. Although the further implications of this difference are not fully described here, one can at least perceive the implication of the entropic index on the thermodynamic variables which -as mentioned before- has great significance in various fields, in particular, High Energy Physics.

4 Conclusion & Summary

In conclusion, we have covered the basic mathematical and conceptual formulation of Tsallis statistics, in particular, the entropy functional form, the interpretation of the entropic index, and the Tsallis probability. Then, we derived the exact formulas for the thermodynamic variables in the case of $m = 0$ and $\mu = 0$ for both quantum and classical particles following Tsallis statistics. In addition, we illustrated the solution of the ideal hydrodynamic equation that is extensively used in High Energy Physics collisions as well as the pressure dependence on temperature and entropic index for protons in the framework of Tsallis and Boltzmann-Gibbs statistics. Finally, these calculations and illustrations emphasize the difference between these two frameworks as well as the wide range of applicability of Tsallis statistics in various fields where non-extensive statistical mechanics is essential, particularly, High Energy Physics.

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