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Numerical methods in theory of topological solitons

PROJECT REPORT
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Contents

—I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

John Scott-Russell

1 Introduction

A soliton, The wave of translation, is a special kind of wave that is characterized by possessing a quantum of energy. They propagate as a traveling wave in nonlinear systems. Solitons are generally described as being localized (finite energy), consistent in structure in absence of interaction terms, and preserved structure after interactions of other waves. They do not obey the superposition principle and does not dissipate.

The study of these special solutions may rather seem peculiar since they arise only nonlinear systems. Nonlinear systems do not rise in elementary studies of newton equation or even quantum mechanics. However, because nonlinear systems are characteristic of nature, these studies may form a concrete understanding of mother nature.Shnir [2018]

Solitons, arising for these nonlinear systems, have these properties due to an astonishing balance between nonlinearity and dissipation in the model. Nonlinearity acts to concentrate the wave further while dissipation spreads the waves apart. Solitons effectively describe a lump of energy of finite mass that propagates in space with no change in structure. This raises the inquiry of whether solitons can effectively describe particles.

1.1 Solitons as particles

In the 1960s various approaches to quantum field theory have been developed. To gain an intuition and prospect candidates of field theory, extensive studies of classical field theories in their fully nonlinear form were made. Requiring that the solution is of finite energy, particle-like solutions were suggested. These solutions have not been recognized before. Different particles arise from the quantization of fields in quantum field theory. Their general characteristics are highly determined by the form of the classical equations.

A feature that characterizes these particle-like solutions are their topological structure. Considering particles that arise in quantum field theory, particles arise from the excitation of the vacuum. Thus, particles in this context represent a deformation of the field which is not generally a topological change. These new particles owe their stability to topological diffesctivness. In the context of these studies these particles generally possessed large energies, however, they could not simply decay into other particles.

The topological character of these solutions can be captured using a single integer Q referred to as the topological charge. Shnir [2018] This is a topological degree of the field. A solution of topological charge Q can be identified with Q particles with energy increases a Q increases. One, then, can establish the energy density of the field. A general feature is that the energy density is smooth and concentrated in some finite regions in space. Such a configuration is referred to as the topological soliton or simply soliton. As common to nomenclature, the ending "-on" particularly indicate a particle-like behavior.

Another interesting characteristic is that of Q less than zero. A typical solution of $Q = -1$ produces a solution of antisoliton. These antisolitons can be annihilated by other solitons or alternately can be pair produced.

Quantization of wave-like field equations gives rise to elementary particle states with the nonlinear terms representing the interaction between particles. These field equations can generally be linearized to give rise to a wave solution. Now, we can investigate solutions to these theories that give rise to solitons. A well-known result of these kinds of linearization is that the interaction energy of the solitons depends on the spatial separation between them. Additionally, the linearized theory can be used to describe the scattering of waves off the soliton, where the equation is linearized around the soliton solution rather than vacuum. This shows that incoming plane waves emerge with the soliton as radially scattered waves. This can be interpreted as,

$$\text{Soliton} + \text{elementary particle} \longrightarrow \text{Soliton} + \text{elementary particle}$$

Furthermore, soliton-soliton scattering produces interesting behavior in linearized field theories. Although the topological charge conservation requires that the solitons would not disappear, some of the kinetic energy can be converted into radiation especially in relativistic collisions. This radiation can be generally described by the linearized wave equations. The interpretation associated with such behavior in quantum field theories is that soliton-soliton collisions are capable of producing elementary particles. Manton and Sutcliffe [2004]

$$\text{Soliton} + \text{Soliton} \longrightarrow \text{Soliton} + \text{Soliton} + \text{elementary particle}$$

The first example of a topological soliton model of a particle was the skyrmion. The skyrmions arose from the Yaskawa model, a theory of spin half nucleons and three types of spinless pions with nucleons interacting through pion exchange. Imposing symmetry arguments on the Lagrangian led to a particular form of the Lagrangian which allowed for the existence of topological solitons. This Skyrmion has rotational degrees of freedom, and Skyrme had the insight to see that when these were quantized it was quite permissible for the state to have spin half. Thus, a purely bosonic theory may give rise to fermionic states.

Numerous work was put forward to study the soliton solution in various field theories. In this report, we study various nonlinear systems for which we search for solitons solutions. We begin by studying kinks produced in classical field theories. In chapter 3, we study the O(3) sigma model which gives rise to energy lumps like soliton solutions using rational maps. We end our discussion by studying rotationally invariant solutions of the Baby skyrmion model. All numerical representations were made using either C++ or Mathematica.

2

Kinks of Classical field Theories

The most elementary topological solitons arise in 1+1 dimensional space that involve a single scalar field. An general Lagrangian of such scalar field can be written as,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi)$$

Using the action principle, we can find the equation of motion as follows,

$$\partial_\mu \partial^\mu \phi + \frac{dU}{d\phi} = 0$$

Using $\mathcal{L} = T - V$, one can then identify the potential energy and kinetic energy as,

$$V = \int_{-\infty}^{\infty} \frac{1}{2}\phi'^2 + U(\phi) dx$$

$$T = \frac{1}{2}\int_{-\infty}^{\infty} \dot{\phi}^2$$

where we identify $\dot{\phi} = \frac{\partial \phi}{\partial t}$ and $\phi' = \frac{\partial \phi}{\partial x}$. Let ν describe the set of vacuum field where, $\nu = \{\phi_0 : \phi_0' = \dot{\phi} = 0, U(\phi_0) = U_{\min}\}$. The existence of topological solitons depends on the existence of multiple vacua. Furthermore, finite energy configurations are classified topologically by elements (ϕ_-, ϕ_+) where,

$$\phi_\pm = \lim_{x \longrightarrow \pm\infty} \phi(x)$$

Solutions that exist between different vacua $\phi_- \neq \phi_+$ are generally known as kinks. Since we are interested in solutions of finite energy, we find the energy functional from the energy-stress tensor produced by invoking Noether's symmetry. The energy stress tensor can be written in the form,

$$T_{\mu\nu} = \left(\frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \right) \partial_\nu \phi - g_{\mu\nu} \mathcal{L} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L}$$

where $g_{\mu\nu}$ is the metric where the field exists. In the context of this chapter we consider $g_{\mu\nu} = \eta_{\mu\nu}$ with signature (+,-). We can now write the energy functional as follows,

$$E = \int_{-\infty}^{\infty} dx T_0^0 = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_t \phi)^2 + \frac{1}{2}(\partial_x \phi)^2 + U(\phi) \right]$$

One can also see that these theories are Lorentz invariant. This generally allows us to study the static solution as an effective solution for the equation. Additionally, if one is particularly interested in investigating the dynamical solution, one can simply Lorentz boost the static configuration. In the static configuration $\partial_t \phi = 0$ which gives an energy functional of the form,

$$\begin{aligned} E &= \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_x \phi)^2 + U(\phi) \right] \\ &= \int_{-\infty}^{\infty} dx \left[\left(\frac{1}{\sqrt{2}} \partial_x \phi \pm \sqrt{U(\phi)} \right)^2 \mp \sqrt{2U(\phi)} \partial_x \phi \right] \end{aligned}$$

One can easily see that the energy is minimal if,

$$\frac{1}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 = U(\phi)$$

One can then introduce a super potential $W(\phi)$ where,

$$\frac{1}{2} \left(\frac{\partial W}{\partial \phi} \right)^2 = U(\phi)$$

Using the above minimization requirement stated above, one can see that the minimum energy bound is satisfied for any energy functional with $E \geq 0$.

$$E \geq \int_{-\infty}^{\infty} dx \sqrt{2U(\phi)} \partial_x \phi$$

Using the super potential term this can be written as,

$$E \geq W(\phi(\infty)) - W(\phi(-\infty))$$

This bound is generally known as Bogomolny bound. This produces a scalar field that stratify the first order equation,

$$\frac{\partial \phi}{\partial x} = \pm \frac{\partial W}{\partial \phi}$$

Additionally, we can define the topological charge associated with the configuration as the integral of space of the zero component of Noether's current. On can find the conserved Noether's current $\partial^\mu J_\mu = 0$ as,

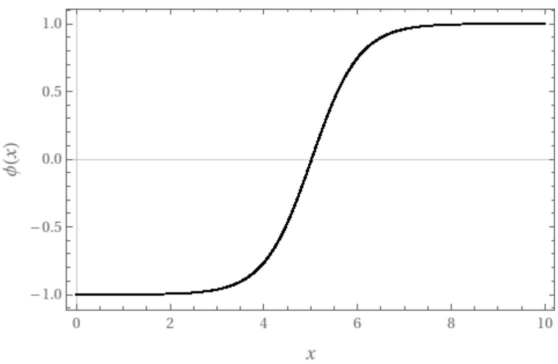
$$J_\mu = \frac{1}{2}\epsilon_{\mu\nu}\partial^\nu \phi$$

where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$ and $\epsilon_{01} = 1$. Thus, The topological charge Q can be defined as,

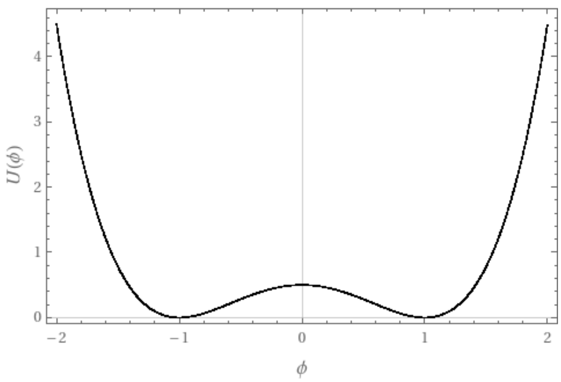
$$Q = \int_{-\infty}^{\infty} dx J_0 = \int_{-\infty}^{\infty} dx \frac{\partial \phi}{\partial x} = [\phi(\infty) - \phi(-\infty)]$$

In this chapter, we study classical kinks associated with 3 different models. Additionally, we develop a numerical algorithm to study the kink antikink collision in the ϕ^4 theory.

2.1 ϕ^4 Theory



(a) Kink Configuration in ϕ^4 theory



(b) Potential ϕ^4 theory

Figure 2.1

This model is the simplest model that posses two vacua. Hassan [2010] This can be easily interpreted from the form of the potential function where we need at least a quadratic term in potential to produce two global minimums as shown in Fig 2.1 (b). We can consider a potential of the form,

$$U(\phi) = \frac{1}{2}(\phi^2 - 1)^2$$

This will produce the following equations of motion,

$$\partial_t^2\phi - \partial_x^2\phi - 2\phi(\phi^2 - 1) = 0$$

Searching for static kink configurations, we need to solve,

$$\partial_x^2\phi = 2\phi(\phi^2 - 1)$$

subject to $\phi(\pm\infty) = \pm 1$. Using the Bogomolny bound,

$$\partial_x\phi = \pm\partial_\phi W = \pm(\phi^2 - 1)$$

Using separation of variables one can find the general solution of the static configuration as,

$$\phi(x) = \tanh(\pm(x - x_0))$$

One can see the symmetry in the produces solutions. The positive solution represent the structure of the kink while its spatial reflection is the antikink configuration. One can also try to approach this problem numerically. To find the solution one can use finite difference method to search for the value of $\phi(i)$ with invoking ± 1 on the boundaries. This would produce configuration as shown in Fig 2.1 (a). Substituting in the equation of topological charge gives $Q = 1$ as expected.

2.1.1 Moving Kinks in ϕ^4 theory

To describe the dynamic of the kinks in ϕ^4 theory, one may just Lorentz boost the static configuration (mix space and time). Thus, using the transformation $x \longrightarrow \gamma(x - vt)$, one can the desired configuration. One can also find these equation by analytically solving the equation of motion.

$$\phi_{tt} - \phi_{xx} + 2\phi(\phi^2 - 1) = 0$$

using a change of variables, let $u = x - vt$ and $\rho = x + vt$,

$$\frac{\partial}{\partial t} = v\left(\frac{\partial}{\partial \rho} - \frac{\partial}{\partial u}\right)$$

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial \rho} + \frac{\partial}{\partial u}\right)$$

substituting back into the equation, we obtain,

$$v^2\left(\frac{\partial}{\partial\rho}-\frac{\partial}{\partial u}\right)\left(\frac{\partial\phi}{\partial\rho}-\frac{\partial\phi}{\partial u}\right)-\left(\frac{\partial}{\partial\rho}+\frac{\partial}{\partial u}\right)\left(\frac{\partial\phi}{\partial\rho}+\frac{\partial\phi}{\partial u}\right)+2\phi(\phi^2-1)=0$$

Assuming $\phi=\phi(x-vt)=\phi(u)$,

$$(v^2-1)\frac{\partial^2\phi}{\partial u^2}+2\phi(\phi^2-1)=0$$

With suitable boundary conditions we can solve this equation analytically. Multiplying the equation by ϕ' where $'$ indicates a derivative with respect to u, we obtain,

$$(v^2-1)\phi'\phi''+2\phi\phi'(\phi^2-1)=0$$

Integrating the equation with respect to u,

$$(v^2-1)\int\phi'\phi''du+2\int\phi\phi'(\phi^2-1)du=C$$

$$\frac{1}{2}(v^2-1)\phi'^2+\frac{1}{2}(\phi^2-1)^2=C$$

Since $\lim_{u\longrightarrow\pm\infty}\phi(u)=\pm1$, $\phi'\longrightarrow0$, We can set $C=0$.

$$(v^2-1)\phi'^2+(\phi^2-1)^2=0$$

Thus, we get this first order DE,

$$\phi'=\frac{\pm1}{\sqrt{1-v^2}}(\phi^2-1)$$

Integrating this equation,

$$\int\frac{d\phi}{\phi^2-1}=\frac{\pm1}{\sqrt{1-v^2}}\int du$$

$$\tanh^{-1}(\phi)=\frac{\mp(u-u_0)}{\sqrt{1-v^2}}$$

Thus,

$$\phi=\tanh\left(\mp\frac{(x-x_0)-v(t-t_0)}{\sqrt{1-v^2}}\right)$$

We can now evaluate the energy and momentum expressions for the moving kink.

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} T_0^0 dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + (1-\phi^2)^2 \\
&= \frac{1}{1-v^2}\int_{-\infty}^{\infty}\operatorname{sech}^4\left(\frac{(x-x_0)-v(t-t_0)}{\sqrt{1-v^2}}\right)dx \\
&= \frac{1}{3\sqrt{1-v^2}}\tanh\left(\frac{(x-x_0)-v(t-t_0)}{\sqrt{1-v^2}}\right)\left[\tanh^2\left(\frac{(x-x_0)-v(t-t_0)}{\sqrt{1-v^2}}\right)-3\right]_{-\infty}^{\infty} \\
&= \frac{4}{3\sqrt{1-v^2}}
\end{aligned}
$$

$$
\begin{aligned}
P &= \int_{-\infty}^{\infty} T_1^0 dx = \int_{-\infty}^{\infty}\partial_x\phi\partial_t\phi dx \\
&= -\frac{v}{1-v^2}\int_{-\infty}^{\infty}\operatorname{sech}^4\left(\frac{(x-x_0)-v(t-t_0)}{\sqrt{1-v^2}}\right)dx \\
&= \frac{v}{3\sqrt{1-v^2}}\tanh\left(\frac{(x-x_0)-v(t-t_0)}{\sqrt{1-v^2}}\right)\left[\tanh^2\left(\frac{(x-x_0)-v(t-t_0)}{\sqrt{1-v^2}}\right)-3\right]_{-\infty}^{\infty} \\
&= \frac{4v}{3\sqrt{1-v^2}}
\end{aligned}
$$

2.1.2 Kink-Antikink collision

An Interesting behavior to investigate is the collision between Kink and Antikink solutions. To describe these collision one can utilize a numerical approach similar to that of Abdelhady.Abdelhady [2012] One can utilize a fourth order finite difference to describe the second spatial derivatives as,

$$\phi_{xx} = \frac{-\phi_{n-2} + 16\phi_{n-1} - 30\phi_n + 16\phi_{n+1} - \phi_{n+2}}{12h^2}$$

where h is the discretize in the spatial coordinate. One then can use a fourth order Rung-Kutta to discretize the time steps. This is done using the following approach. First we utilize a shooting method to break the second partial derivative of time as,

$$\frac{\partial\phi}{\partial t} = v = f_\phi(\phi, v, x)$$

$$\frac{\partial v}{\partial t} = \phi_{xx} - 2\phi(\phi^2 - 1) = g_v(\phi, v, x)$$

Using a initial condition of ϕ_0 and v_0 the describe the one kink and one antikink spatially separated and their time derivatives,

$$\phi_0 = \tanh\left(\frac{x-x_0}{\sqrt{1-v^2}}\right) + \tanh\left(-\frac{x+x_0}{\sqrt{1-v^2}}\right) + 1$$

$$v_0 = \frac{v}{\sqrt{1-v^2}}\operatorname{sech}^4\left(\frac{x-x_0}{\sqrt{1-v^2}}\right) + \frac{v}{\sqrt{1-v^2}}\operatorname{sech}^4\left(-\frac{x+x_0}{\sqrt{1-v^2}}\right)$$

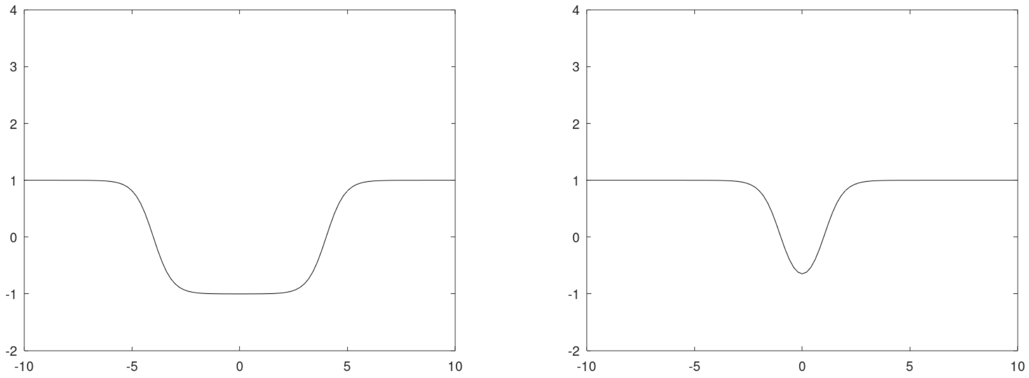
These initial conditions describe a kink moving to the left and an antikink moving right. We now follow this discretization scheme for every time step,

$$\begin{aligned}
k_{1\phi} &= f_\phi(\phi_0, v_0, x_0) = v_0 \\
k_{1v} &= g_v(\phi_0, v_0, x_0) = \phi_{0,xx} - 2\phi_0(\phi_0^2 - 1)^2 \\
\phi_1 &= \phi_0 + \frac{1}{2}k_{1\phi}dt \\
v_1 &= v_0 + \frac{1}{2}k_{1v}dt \\
k_{2\phi} &= f_\phi(\phi_1, v_1, x_1) = v_1 \\
k_{2v} &= g_v(\phi_1, v_1, x_1) = \phi_{1,xx} - 2\phi_1(\phi_1^2 - 1)^2 \\
\phi_2 &= \phi_1 + \frac{1}{2}k_{2\phi}dt \\
v_2 &= v_1 + \frac{1}{2}k_{2v}dt \\
k_{3\phi} &= f_\phi(\phi_2, v_2, x_2) = v_2 \\
k_{3v} &= g_v(\phi_2, v_2, x_2) = \phi_{2,xx} - 2\phi_2(\phi_2^2 - 1)^2 \\
\phi_3 &= \phi_2 + k_{3\phi}dt \\
v_3 &= v_2 + k_{3v}dt \\
k_{4\phi} &= f_\phi(\phi_3, v_3, x_3) = v_3 \\
k_{4v} &= g_v(\phi_3, v_3, x_3) = \phi_{3,xx} - 2\phi_3(\phi_3^2 - 1)^2
\end{aligned}$$

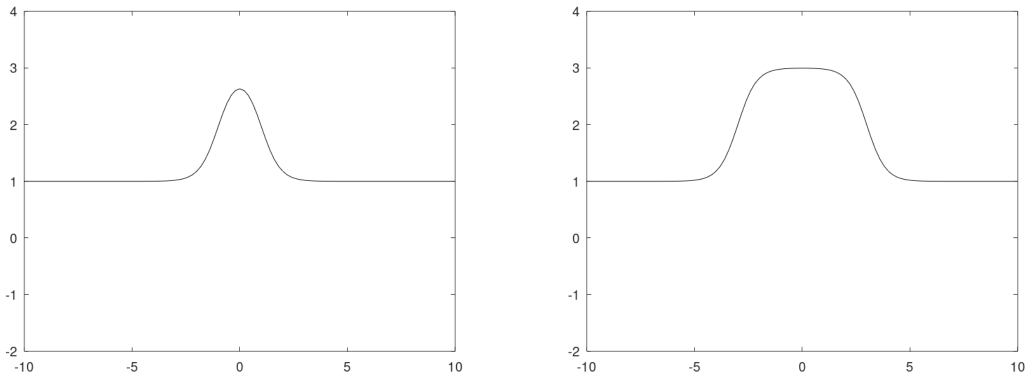
where each of these equation runs all over the spatial points. From these weight factors, we can find the next step of time by,

$$\begin{aligned}
v^{i+1} &= v^i + \frac{dt}{6}\left(k_{1v} + 2k_{2v} + 2k_{3v} + k_{4v}\right) \\
\phi^{i+1} &= \phi^i + \frac{dt}{6}\left(\phi_{1v} + 2\phi_{2v} + 2\phi_{3v} + \phi_{4v}\right)
\end{aligned}$$

The collision annihilates the kinks and antikink pair and pair produces two other configurations moving the same direction as before. Fig 2.2 shows the produced results.



(a) Configuration at 100 time steps (b) Configuration at 400 time steps



(c) Configuration at 600 time steps (d) Configuration at 800 time steps

Figure 2.2: Figures a-b show the interaction between a kink moving to the left with an antikink moving to the right.

2.2 ϕ^6 Theory

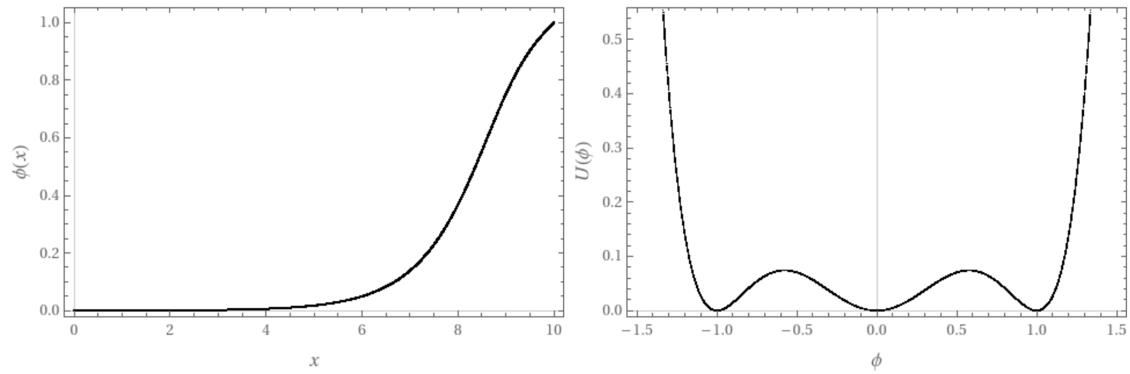
Now let's consider a potential of the form,

$$U(\phi) = \frac{1}{2}\phi^2(1-\phi^2)^2$$

As clearly seen, this potential has three vacua at (-1,0,1). Fig 2.3 show the plot of the corresponding potential. The corresponding equation of motion is,

$$\partial_\mu \partial^\mu \phi = 4\phi^4 - 3\phi^5$$

As discussed before to produce a kink solution, we need the potential to reach the vacuum solution at spatial infinities. In this model, can we effectively construct



(a) Kink (0,1) Configuration in ϕ^6 theory (b) Potential ϕ^6 theory

Figure 2.3

two different kinks by requiring,

$$\phi(\infty) = 1, \qquad \phi(-\infty) = 0$$

or

$$\phi(\infty) = 0, \qquad \phi(-\infty) = -1$$

One can then use the Bolognomy equation to find the a solution for the model.

$$\partial_x \phi = \pm\phi(1 - \phi^2)$$

Using separation of variable, one can find 4 solutions to the equations, but using the boundary conditions two of them are satisfied,

$$\phi(x) = \pm\frac{1}{\sqrt{1 + e^{\mp 2(x-x_0)}}}$$

One can also solve find the solutions numerically using finite difference method. Fig 2.3 show the solution between vacua (0,1).

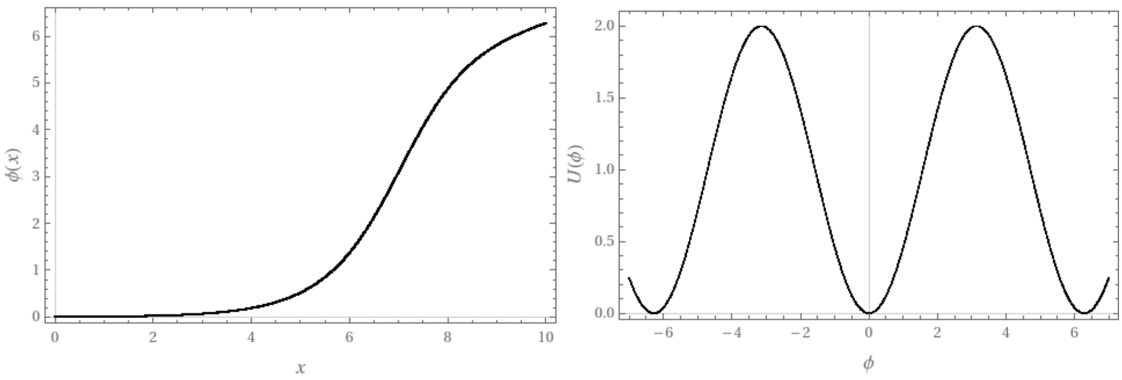
2.3 Sine-Gorden model

Another interesting model to investigate is the sine-Gordon model. The sine-Gordon model has the corresponding potential,

$$U(\phi) = 1 - \cos(\phi)$$

One can then find this equation of motion,

$$\partial_\mu \partial^\mu \phi = -\sin(\phi)$$



(a) Kink (0,1) in Sine-Grodon theory (b) Potential Sine-Grodon theory

Figure 2.4

This model is particularly interesting since there are infinite vacua with period 2π. We can choose the two vacua $(0,2\pi)$ to model the equation. This equation has an analytical solution similar to the previous cases,

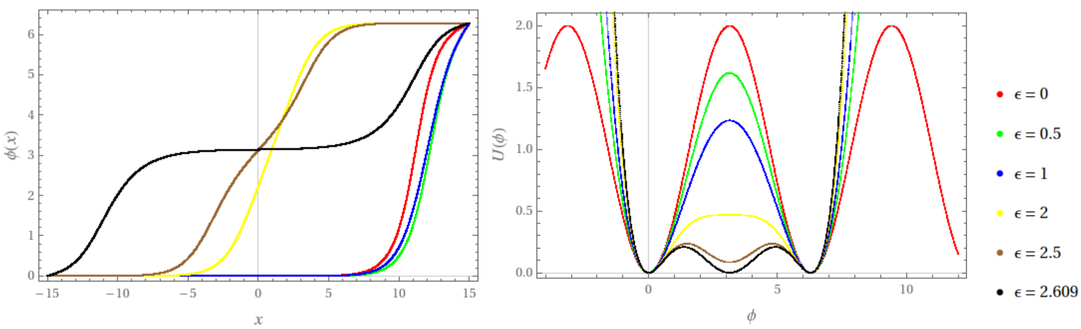
$$\partial_x \phi = \pm\sqrt{2(1-\cos(\phi))} = \pm 2\sin(\frac{\phi}{2})$$

which gives a solution of,

$$\phi(x) = \pm 4\tan^{-1}(e^{x-x_0})$$

Additionally, the numerical results are show in Fig 2.4.

2.4 Parameterized Potential



(a) Kink parameterized theory (b) Potential parameterized theory

Figure 2.5

Let us now consider a case of the combination of the two models. We can construct this model using a parameter ϵ as follows,

$$U(\phi) = (1-\epsilon)(1-\cos(\phi)) + \frac{\epsilon\phi^2}{8\pi^2}(\phi - 2\pi)^2$$

One can see that this model has two degenerate vacua at 2π and 0. Using the previous procedure we can find the equation of motion as,

$$\partial_\mu\phi\partial^\mu\phi = -(1-\epsilon)\sin(\phi) + \frac{\epsilon\phi}{2\pi^2}(\phi - 2\pi)(\phi - \pi)$$

The static equation cannot be solved analytically. Thus, one can only investigate the numerical solution. Fig 2.5 shows different plots for different ϵ with their corresponding potential.

3

O(3) Nonlinear Sigma Model

Another simple model to construct solitons is the O(3) model. However, being dynamical unstable which makes them non soliton solutions, they are worth studying. A sigma model is a nonlinear scalar field theory where the field takes values is a target space which is curved Riemannian manifold. In this chapter, we will study the simplest sigma model with the target space being a 2 sphere, S^2. To construct such model one has to have a 3 component field construed on the surface of the sphere. A Lagrangian which formulates such model in (d+1) dimensions is,

$$\mathcal{L} = \frac{1}{4}\partial_\mu \phi \cdot \partial^\mu \phi + \lambda(1 - \phi \cdot \phi)$$

with the second term representing the constraint with Lagrange multiplier λ. Variation of the Lagrangian will result in the following equation of motion,

$$\partial_\mu \partial^\mu \phi^a + \lambda \phi^a = 0$$

using $\phi^a \cdot \phi^a = 1$, we can find λ as,

$$\lambda = -\phi^a \partial_\mu \partial^\mu \phi^a$$

Substituting this into the above equation we get,

$$\partial_\mu \partial^\mu \phi^a - \phi^a(\phi^b \partial_\mu \partial^\mu \phi^b) = 0$$

Since,

$$\partial_\mu(\phi^a \partial^\mu \phi^a) = \partial_\mu \phi^a \cdot \partial^\mu \phi^a + \phi^a \partial_\mu \partial^\mu \phi^a = \frac{1}{2}\partial_\mu \partial^\mu(\phi^a \cdot \phi^a) = 0$$

Thus, the equation of motion for the O(3) model has the form,

$$\partial_\mu \partial^\mu \phi^a + \phi^a(\partial_\mu \phi^b \cdot \partial^\mu \phi^b) = 0$$

3.1 Derrick's Theorem

As previously discussed in the last chapter, The energy density associated with a Lagrangian of the form $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - U(\phi)$ is,

$$E = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + U(\phi)$$

A generalization of this energy functional which contains as set of scalar fields $\vec{\phi} = (\phi^1, \phi^2, \cdots, \phi^n)$ can be written in the form,

$$E = \int_{-\infty}^{\infty} d\vec{x} \left[\frac{1}{2}\partial_\mu \vec{\phi} \partial^\mu \vec{\phi} + U(\vec{\phi})\right]$$

Since, we are searching for soliton solutions, we require the potential to vanish in state ϕ_0. Additionally, by definition, solitons are localized that makes them have a consistent form. This is referred to as the characteristic size of the solitons R. By this construction, the global minimum of the functional should not be affected by any transformations $x \longrightarrow \lambda x$. with these transformations, we expect the derivative to transform inverse of x with Jacobin λ^{-n}, where n represents the dimension of space.

$$E \longrightarrow \lambda^{2-n}\frac{1}{2}\int_{-\infty}^{\infty} d\vec{x}\partial_\mu \vec{\phi} \partial^\mu \vec{\phi} + \lambda^{-n}\int_{-\infty}^{\infty} d\vec{x} U(\vec{\phi})$$

We can thus rename the integral by E_2 and E_0 respectively. Let us now define a parametric family of the field configuration $\phi_\lambda(x) = \phi(\lambda x)$. We can, thus, find the minimum of the the scaled energy functional by differentiating with respect to λ.

$$\frac{dE}{d\lambda} = (2-n)\lambda^{1-n}E_2 - n\lambda^{-n-1}E_0 = 0$$

From this, we can find the conditions that of existence and stability of the energy functional for $\lambda = 1$. As it can be seen that the existence of a stationary point depends of the dimensionality of the model. For one dimension model, a stability exist with deformations, $\lambda = \sqrt{E_0/E_2}$ which are the cases of ϕ^4 and sine Gordon models.For $n = 2$, E_0 must be equal to zero, which requires $\vec{\phi}$ to always exist in the vacuum state. However, this does not prevent the existence of solitons, as we will see in this chapter.

Furthermore, we can check the stability of the points by finding the second derivative for various cases.

$$n = 1 \qquad \frac{d^2E}{d\lambda^2}\mid_{\lambda=1} = 2E_0 > 0 \qquad \text{minimum point}$$

$$n = 2 \qquad \frac{d^2E}{d\lambda^2}\mid_{\lambda=1} = 6E_0 = 0 \qquad \text{no minimum}$$

$$n > 2 \qquad \frac{d^2E}{d\lambda^2}\mid_{\lambda=1} = (2-n)(1-n)E_2 + n(n+1)E_0 = 2nE_0 < 0$$

Thus, the Derrick's Theorem predicts that there is no soliton solution to exist in $n > 2$ in a certain model since the energy functional has no stable minimum. There are methods to evade such conditions that will be discussed later in chapter 4.

3.2 O(3) Model verses $\mathbb{C}P^1$

In the realm of Derricks theorem, we seek clues to construct solitons configurations in 2+1 spaces. As it can be seen that if we restrict our discussion to scalar fields instability in the configuration will arise. Thus, one has to break the scale invariance of the model. O(3) are the present solutions to such problems. It was found that upon discussing a target space $\mathcal{M} = S^{2N}$, the coset description of the manifold yields an isomorphism between the target space and the complex plane. Thus,

$$\mathcal{M} = \frac{SU(N+1)}{SU(N) \times U(1)} = \mathbb{C}P^N$$

As previously mentioned, we specifically target the manifold S^2, thus, we will be working with $\mathbb{C}P^1$ to solve field equations. Since our goal is to construct finite energy configurations, we can find the finite energy of the static configuration as,

$$E = \frac{1}{4}\int_{-\infty}^{\infty} d^2x \partial_\mu \phi \cdot \partial_\mu \phi$$

For E to be finite $\vec{\phi}$ must be a constant vector at spatial infinity. Without any loss of generality we can take, $\phi^\infty = (0, 0, 1)$. We can take this to correspond to the minimum of the energy functional. This yield the compaction of $\mathbb{R}^2$ (or $\mathbb{C}$) to S^2 which makes the associated field of the O(3) a map $\phi : S^2 \mapsto S^2$, from the physical field to the target space. This classified by a homotopy group $\pi_2(S^2) = \mathbb{Z}$. This permits the existence of topologically non trivial soliton solutions with topological charge Q. An explicit representation to the topological charge can be written using an area element of the target space as,

$$Q = \frac{1}{4\pi}\int d^2\vec{x}\vec{\phi}\cdot\left[\frac{\partial\vec{\phi}}{\partial x}\times\frac{\partial\vec{\phi}}{\partial y}\right] = \frac{1}{8\pi}\int d^2x\epsilon_{abc}\epsilon_{ij}\phi^a\partial_i\phi^b\partial_j\phi^c$$

Additionally to find lower energy bounds for this configuration, we can consider the auxiliary quantity,

$$F_i^a = \partial_i\phi^a \pm \epsilon_{abc}\epsilon_{ij}\phi^b\partial_j\phi^c$$

Since,

$$\frac{1}{2}\int d^2x(F_i^a)^2 = \int d^2x\left[\partial_i\phi^a \mp \epsilon_{abc}\epsilon_{ij}\phi^a\partial_i\phi^b\partial_j\phi^c\right] \geq 0$$

This is known as the Belavin–Polyakov inequality, which gives the following result,

$$E \geq 4\pi Q$$

with the bound being $E = 4\pi Q$. To effectively analyze the Belavin–Polyakov bound, one utilizes the an effective change of variables,

$$W = \frac{\phi_1 + i\phi_2}{1-\phi_3}$$

where W and $\bar{W}$ are inhomogeneous projections of the 2 sphere to the $\mathbb{C}P^1$. To find the stereographic projection from the North pole, one needs to connect a line from the north pole (0,0,1) to the point $(u, w, 0)$ that passes through the point (ϕ_1, ϕ_2, ϕ_3). Projecting this line on the planes ϕ_1-ϕ_2, ϕ_2-ϕ_3 and ϕ_1-ϕ_3, one can see that the following relations must hold for the line to pass through the mentioned points.

$$u : w : -1 = \phi_1 : \phi_2 : \phi_3 - 1$$

These relations are basically the equations of straight lines in each plane that comes from subsequent subtraction of given points. One can then deduce the following equations for u and w,

$$u = \frac{\phi_1}{1-\phi_3}, \qquad w = \frac{\phi_2}{1-\phi_3} \tag{3.1}$$

using $\phi_1^2 + \phi_2^2 + \phi_3^2 = 1$, one can find, by squaring the above relations that,

$$\begin{aligned} u^2 + w^2 &= \frac{1-\phi_3^2}{(1-\phi_3)^2} \\ &= -1 + \frac{2}{1-\phi_3} \end{aligned}$$

or,

$$u^2 + w^2 + 1 = \frac{2}{1-\phi_3} \tag{3.2}$$

Thus, we can find ϕ_3 as follows,

$$\phi_3 = \frac{u^2 + w^2 - 1}{u^2 + w^2 + 1}$$

Additionally, by substituting equation 2 into equations 1 for $(1 - \phi_3)$, one can find the whole set of inverse transformations given be,

$$(\phi_1, \phi_2, \phi_3) = \left(\frac{2u}{u^2 + w^2 + 1}, \frac{2w}{u^2 + w^2 + 1}, \frac{u^2 + w^2 - 1}{u^2 + w^2 + 1} \right)$$

Furthermore, one can find subsequent projections from the south pole $(0, 0, -1)$ with point $(u', w', 0)$. Using the same argument as above, one can see that,

$$u' = \frac{\phi_1}{1 + \phi_3}, \qquad w' = \frac{\phi_2}{1 + \phi_3} \tag{3.3}$$

with inverse transformations,

$$(\phi_1, \phi_2, \phi_3) = \left(\frac{2u^{'}}{u'^2 + w'^2 + 1}, \frac{2w^{'}}{u'^2 + w'^2 + 1}, \frac{1 - u^{'2} - w^{'2}}{u'^2 + w'2 + 1} \right)$$

Using this transformation, we are able to effectively write the Lagrangian, energy density and topological charge as follows,

$$\mathcal{L} = \frac{\partial_\mu W \partial^\mu \bar{W}}{(1 + |W|^2)^2} \tag{3.4}$$

$$E = 2 \int d^2x \frac{|\partial_z W|^2 + |\partial_{\bar{z}} W|^2}{(1 + |W|^2)^2} \tag{3.5}$$

$$Q = \frac{1}{\pi} \int d^2x \frac{|\partial_z W|^2 - |\partial_{\bar{z}} W|^2}{(1 + |W|^2)^2} \tag{3.6}$$

where $\partial_z = \frac{1}{2}(\partial_i - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_i + i\partial_y)$. One can also attempt to find equivalent map from the South Stereographic project that will leave the energy density invariant. As a motivation from the given rational map, $W = (\phi_1 + i\phi_2)/(1 - \phi_3) = u + iw$ for configuration of $Q = 1$ soliton from the north stereographic projection, One can try to find relations between (u, w) and (u', w') in order to construct an equivalent rational map from the south stereographic projection. This equivalent has also to satisfy the same energy distribution relation (i.e leaves E invariant under transformation $W_N \longrightarrow W_S$). Using equations 1 and 2 one can easily see that,

$$\frac{u'}{w'} = \frac{u}{w} \tag{3.7}$$

$$uu' + ww' = \frac{\phi_1^2 + \phi_2^2}{1 - \phi_3^2} = 1 \tag{3.8}$$

Factorizing the latter term ad noticing that $W_N = u + iw$,

$$\begin{aligned}
1 &= uu' + ww' \\
&= (u + iw)(u' - iw') - iwu' + iw'u \\
&= (u + iw)(u' - iw') \\
&= W_N W_S
\end{aligned}$$

Line 3 utilized equation 4. Line 4 called the right term W_S. This naming is due to the fact that E is invariant under transformation $W \longrightarrow 1/W$. We can see this using the formula for static energy density, we can always do this transformation leaving the energy density invariant, but this is just the definition for W_S we have previously assumed. Thus, one can find the equivalent rational map for the south stereographic projection just be inverting the north rational map.

Another interesting consequence for this change of variables can be found by apply Bogomolny equation to equations 3.5 and 3.6. A form equivalent to the Cauchy remain conditions is found,

$$\partial_{\bar{z}} W = 0$$

or for the minus sign

$$\partial_z W = 0$$

which make W either a holomorphic or antiholomorphic function that satisfy Bogomolny equation. Since W is a map to a Riemannian sphere, it is acceptable for W to have poles. Thus, for example, if $W(z_0)$ is singular it will directly transformed to $(0, 0, -1)$ with no effect on the energy density or the topological charge. Additionally, we require $W(z)$ as z approaches ∞ to be finite for the total energy to be finite. This clearly indicates that $R(z)$ can be written as a rational function of z,

$$R(z) = \frac{p(z)}{q(z)}$$

where we identify the topological charge as being equal to the topological degree of the rational map $Q = \max\{\deg(p), \deg(q)\}$. Manton and Sutcliffe [2004] The solutions are required to satisfy the boundary condition at spatial infinity $\phi(\infty) = \phi^{\infty}$, or for the rational map $W(\infty) = 0$. For this to happen, $\deg(p(z))$ must be less than $\deg(q(z))$.

3.3 Topological Soliton Solutions

In this section, we will be investigating a set of rational maps to construct a set of soliton solutions. In what follows, we use Mathematica as an effective symbolic manipulation tool.

3.3.1 Q=2 Configuration

Consider a rational map of the form,

$$W(z) = \frac{(z-a)(z-b)}{(z-c)(z-d)}$$

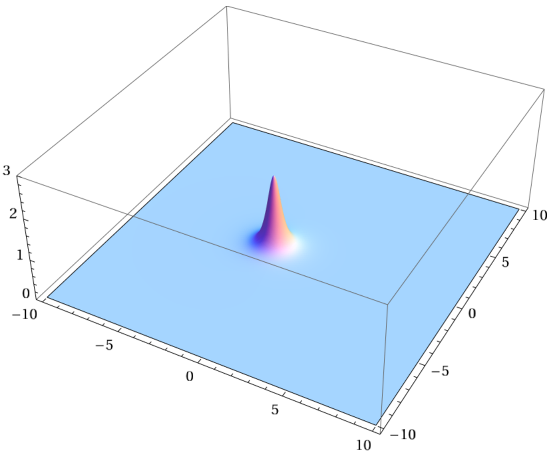
where a,b,c and d $\in \mathbb{C}$. This configuration corresponds to two lumps. To find the energy density we use equation 3.5 and plot the function for different values of a,b,c, and d. Fig 3.1 shows the results of with the parameter specified.

3.3.2 Q=8 Configuration - Pyramid

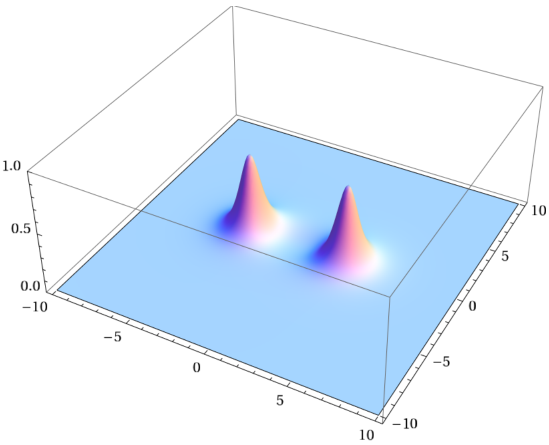
A configuration of $Q = 8$ can be made using the following rational map.

$$W(z) = \frac{4}{\frac{1}{z} + \frac{1}{z+0.5-i} + \frac{1}{z-0.5-i} + \frac{1}{z-1} + \frac{1}{z+1} + \frac{1}{z+1.5+i} + \frac{1}{z-1.5+i} + \frac{1}{z-2i}}$$

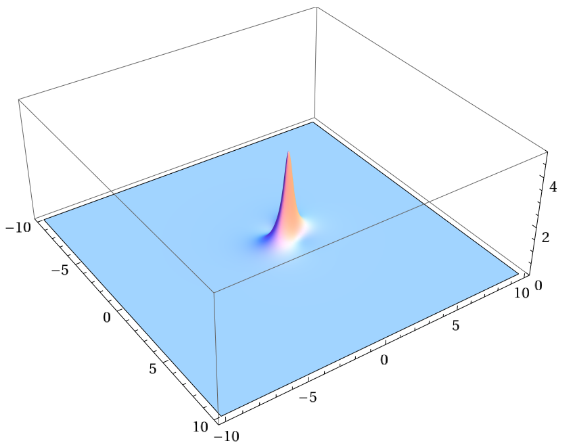
This configuration will effectively build structures in the form of a pyramid. Fig 3.2 shows the energy density plot.



(a) a=0.5, b=2, c=-0.5, d=0.5



(b) a=1, b=4, c=-1, d=-4



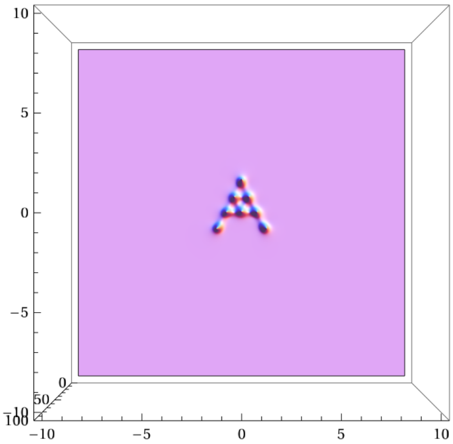
(c) a = 2, b = 2i, c = 1+i, d = -1+i

Figure 3.1: Energy density for soliton configuration $Q = 2$ with various parameter values.

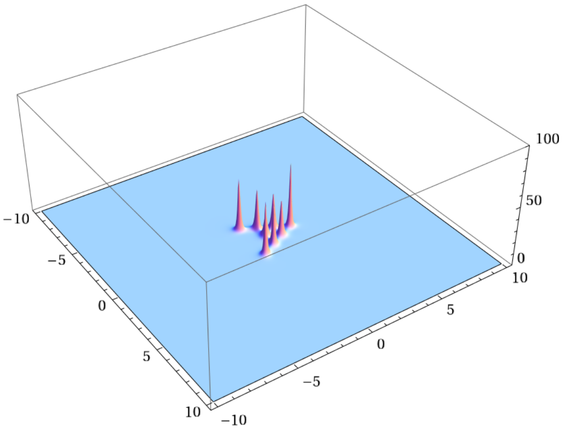
3.3.3 Q=8 Configuration - Line

Either by directly evaluating maximums of the energy density functional or by effective realization of the position of $Q = 1$ in the rational map, $W(z) = \lambda e^{i\theta}/(z - a_1)$, one can deduce that a_1 is the maximum of the energy density. Additionally, one can also find the map that produces a configuration of 8 solitons along the x-axis by noting that the constants beside the z variables are just the maximums of the energy function. Thus, the map will have the following form,

$$W(z) = \frac{4}{\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z-1} + \frac{1}{z-2} + \frac{1}{z+2} + \frac{1}{z+3} + \frac{1}{z-3} + \frac{1}{z-4}}$$

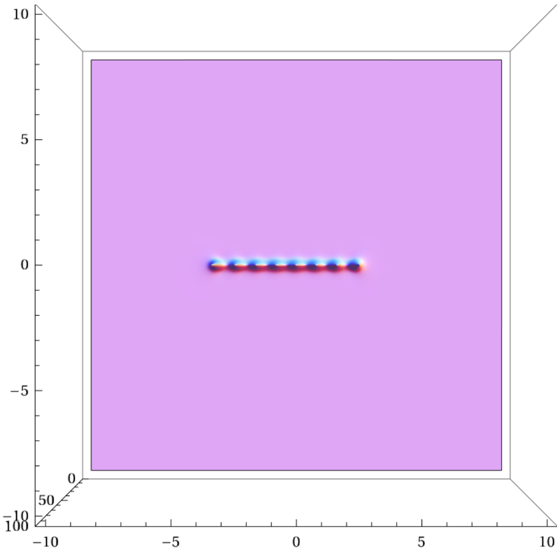


(a) Top view

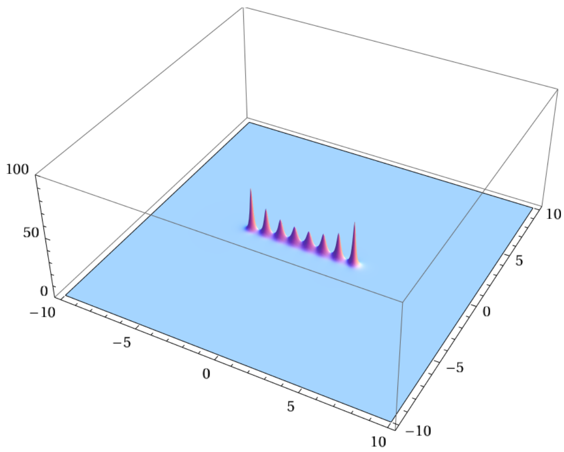


(b) Tilted view

Figure 3.2: Energy density for soliton configuration $Q = 8$ pyramid structure



(a) Top view



(b) Tilted view

Figure 3.3: Energy density for soliton configuration $Q = 8$ line structure

The results are shown in Fig 3.3.

4
Baby Skyrmions

Let us now return to the problem of constructing models that admits solitons in higher dimensions. As previously discussed, according to Derricks theorem, for $n > 2$ there is no static, finite energy solutions for the model $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$. The problem for higher dimensional theories arises because the theories are scale invariant. However, there are methods to overcome such a problem. A given solution is proposed by introducing interaction through gauge theory and localize the scale symmetry. We can couple the scalar field with an abelian gauge field by introducing gauge $A_\mu(x)$ and changing the usual derivative to a covariant derivative

$$D_\mu\vec{\phi} = \partial_\mu\vec{\phi} + gA_\mu(x)\vec{\phi}$$

where g is the coupling constant. One can, then, find the gauge field by requiring that the covariant derivative transforms as the field,

$$D_\mu\vec{\phi}(x) \longrightarrow \lambda D_{\mu'}\vec{\phi}(x')$$

Clearly, we can see that the field transforms as $A_\mu(x) \longrightarrow \lambda A_\mu(x')$. Thus, the field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

transforms as,

$$F_{\mu\nu}(x) \longrightarrow \lambda^2 F_{\mu\nu}(x')$$

Thus, the new Lagrangian can be written as,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - U(\phi)$$

Thus, the energy would have an additional term that scale as λ^4 that we donate as E_4,

$$E_4 = \frac{1}{4}\int d^n x F^{\mu\nu}F_{\mu\nu}$$

Thus, the total energy scales as,

$$E = E_0 + E_2 + E_4 \longrightarrow \lambda^{-n}E_0 + \lambda^{2-n}E_2 + \lambda^{4-n}E_4$$

Again, we differentiate with respect to λ and then substitute $\lambda = 1$. Thus,

$$\frac{dE}{d\lambda} = -n\lambda^{-n-1}E_0 + (2-n)\lambda^{1-n}E_2 + (4-n)\lambda^{3-n}E_4 = 0$$

Thus, for $n = 2$ there is stability if $E_4 = E_0$. Relations of this type are known as virial relations. For $n = 3$ systems, the virial relations are $E_4 = E_2 + 3E_0$. These are called Hooft–Polyakov monopoles. Lastly, when $n = 4$, the scale invariance is restored and we have $E_0 = E_4 = 0$ with only vacuum solutions in the pure gauge sector are allowed.

We now consider such model with interaction termed as the Baby Skyrme model. The Lagrangian in 2+1 dimension of the model is,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{1}{4}(\epsilon_{abc}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c)^2 - U(|\phi|)$$

where $a = 1, 2, 3$. Additionally, the scalar field components are constrained on the surface of a sphere $\phi^a \cdot \phi^a = 1$. So, the target space is S^2 similar to the O(3) model. This also makes the baby Skyrme model support a soliton solution with a topological charge,

$$Q = \frac{1}{8\pi}\int d^2x \epsilon_{abc}\epsilon_{ij}\phi^a\partial_i\phi^b\partial_j\phi^c$$

with an energy functional of the form,

$$E = \int d^2x \left[\frac{1}{2}(\partial_i\phi^a)^2 + \frac{1}{4}(\epsilon_{abc}\phi^a\partial_i\phi^b\partial_j\phi^c)^2 + U(|\phi|)\right]$$

The presence of the Skyrme term breaks the scale invariance, however, the potential term is necessary for the energy functional to posses a minimum. A common choice for the potential is,

$$U(\phi) = \mu^2(1 - \phi_3)$$

where μ is the mass parameter. This potential breaks the O(3) symmetry to O(2) with a single vacuum at $\phi_3 = 1$. An interesting way of deriving the energy bound of the Skyrme model is by using the strain tensor.

$$D_{ij} = \partial_i\phi^a\partial_j\phi^a$$

where, D_{ij} has non negative eigenvalues λ_1^2, λ_2^2. One can see that using the simple relation discussed in the beginning of chapter 3 that the determinant of the stress tensor has the form,

$$\det(D) = (\epsilon_{abc}\phi^a\partial_i\phi^b\partial_j\phi^c)^2 = \lambda_1^2\lambda_2^2$$

and

$$\mathrm{Tr}(D) = (\partial_i\phi^a)^2 = \lambda_1^2 + \lambda_2^2$$

Thus, we can write the energy functional and topological charge as,

$$E = \int d^2x \left[\frac{1}{2}(\lambda_1^2 + \lambda_2^2) + \frac{1}{2}\lambda_1^2\lambda_2^2 + U(\phi)\right]$$

$$Q = \frac{1}{4\pi}\int d^2\lambda_1\lambda_2 x$$

We can then see that,

$$\frac{1}{2}\int d^2x\left[(\lambda_1 \pm \lambda_2)^2 \mp 2\lambda_1\lambda_2\right] = \frac{1}{2}\int d^2x(\lambda_1^2 + \lambda_2^2)^2 \geq \left|\int d^2\lambda_1\lambda_2 x\right| = 4\pi Q$$

Thus, we cannot reach the topological bound which prevents us from using Bogomolny equation. Following a similar approach as in the previous chapter by introducing the constant $\phi^a \cdot \phi^a = 1$ with a Lagrange multiplier λ, then effectively compensate for the multiplier we reach an equation of the form,

$$\begin{aligned}
&\partial_\mu\partial^\mu\phi^a - \phi^a(\phi^b\partial_\mu\partial^\mu\phi^b) - (\partial_\nu\phi^b\partial^\nu\phi^b)\partial_\mu\partial^\mu\phi^a \\
&\quad - (\partial_\mu\partial^\nu\phi^b\partial^m u\phi^b)\partial_\nu\phi^a + (\partial^\nu\phi^b\partial^\nu\phi^b)\partial_\nu\partial^\mu\phi^a + (\partial_\nu\partial^\nu\phi^b\partial^m u\phi^b)\partial_\mu\phi^a \\
&\quad - (\partial_\mu\phi^b\partial^\mu\phi^b)(\partial_\nu\phi^c\partial^\nu\phi^c)\phi^a + (\partial_\nu\phi^b\partial^\mu\phi^b)(\partial^\nu\phi^c\partial^\mu\phi^c)\phi^a + U'\phi^a
\end{aligned} \tag{4.1}$$

4.1 Soliton Solutions

This equation is highly nonlinear which has a solution only numerically. However, for $Q = 1$ we can simplify this relation. We expect that the solution to be spherically symmetric, thus, we can consider the hedgehog anataz,

$$\phi_1 = \cos\theta\sin f(r), \qquad \phi_2 = \sin\theta\sin f(r), \qquad \phi_3 = \cos f(r)$$

$f(r)$ is a decreasing function r of to make the field reach the vacuum solution as $r \longrightarrow \infty$. Substituting this result in equations for energy density and topological charge, Shnir [2018]

$$E = 2\pi\int_0^\infty rdr\left[\frac{1}{2}f'^2 + \frac{\sin^2 f}{2r^2}(f'^2 + 1) + \mu^2(1 - \cos f)\right]$$

$$Q = -\frac{1}{2}\int_0^\infty r dr \frac{f' \sin f}{r} = \frac{1}{2}(\cos f(\infty) - \cos f(0))$$

For, $Q = 1$ we must have $f(0) = \pi$ and $f(\infty) = 0$. One can also find the form of the variational equation after substitution in equation 4.1,

$$\left(r + \frac{\sin^2 f}{r}\right) f'' + \left(1 - \frac{\sin^2 f}{r^2} + \frac{f' \sin f \cos f}{r}\right) f' - \frac{f' \sin f \cos f}{r} - r\mu^2 \sin f = 0 \tag{4.2}$$

Using equation 4.2 and boundary conditions specified above we find a numerical solution for the form of f. Additionally, we can use this results and substitute for the energy functional. Fig 4.1 shows the corresponding results.

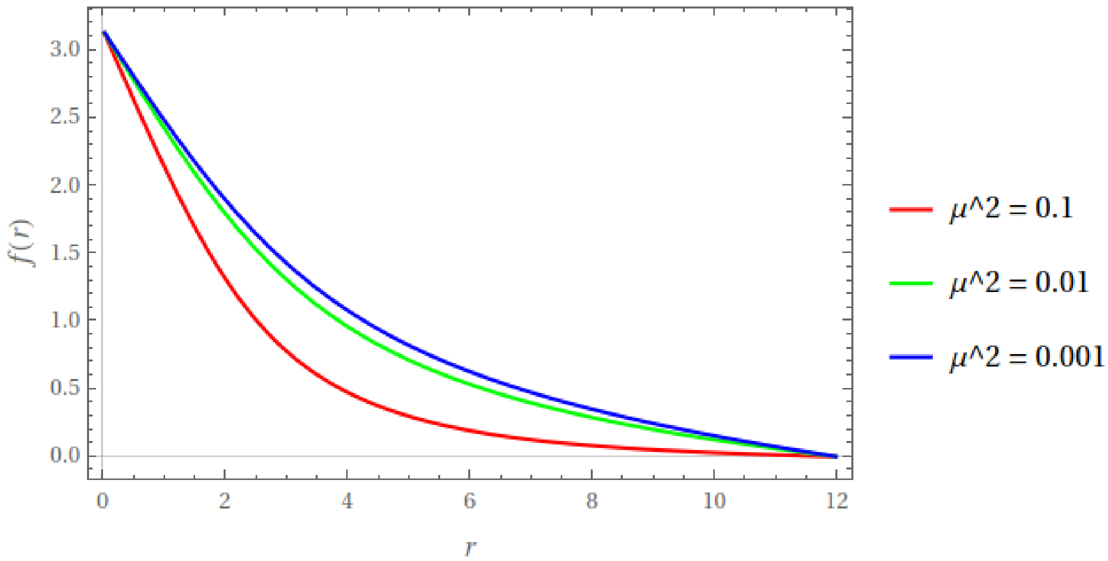


Figure 4.1: $f(r)$ numerical solution for different values of μ^2

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