Puzzles of Multiplicity

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Contents

Abstract

Multiparticle production lies in the core of relativistic quantum mechanics and forms a key component of our understanding of high–energy interactions. This project focuses on the study of quark and gluon production and their subsequent transformation into hadrons through the process of hadronization. The entire sequence of events can be effectively described using a two-stage model, supported by a Markov-chain branching mechanism. In this report, we outline the theoretical framework of these models and comparision with experimental and present the corresponding calculations and results.

1 Introduction

Multiparton spectra in QCD, especially those characterizing quark and gluon jets, play a central role in understanding high-energy particle production. The evolution of these jets can be described through elementary QCD branching processes such as gluon fission ($g \to gg$), quark bremsstrahlung ($q \to qg$), and quark–pair creation ($g \to q\bar{q}$). These processes generate rich quark-cascade remineds electromagnetic showers in QED, but gluons can interact between themselves while forbidding quark–pair creation reproduces classical ϕ^3-theory results for gluon distributions. Modeled via Markovian branching and nonlinear generating-function equations, this framework provides a robust and transparent description of parton multiplication, enabling precise predictions for jet multiplicities, correlations, and hadronization patterns observed in high-energy collisions. .This reports includes experimental values fitting with theoretical values of 14GeV, 22GeV, 34GeV and 50GeV energies.

2 Two Stage Model

The Gluon Dominance Model describes multiplicity distributions, it consists of two stages. The first stage treats the perturbative QCD parton cascade, while the second (hadronization) stage is described by a phenomenological model. The simplest case for study is electron–positron annihilation, which exhibits a quark–gluon cascade in which quarks radiate gluons, gluons undergo fission, and eventually all partons convert into hadrons. Theoretical investigation and experimental data indicate that bremsstrahlung gluons dominate the high-multiplicity behaviour: the valence quarks remain in leading particles and act passively, whereas gluons largely generate the large final-state multiplicity.

The elementary QCD branchings relevant to the cascade are

1. gluon fission: $g \to g + g$,
2. quark bremsstrahlung: $q \to q + g$,
3. quark–pair creation: $g \to q + \bar{q}$.

It is convenient to work with generating functions. For a single variable multiplicity generating function one writes

$$Q(s,z) \;=\; \sum_{n=0}^{\infty} P_n(s)\, z^n,$$

so that the multiplicity distribution and factorial moments follow from derivatives of Q:

$$P_n(s) = \frac{1}{n!}\,\frac{\partial^n Q(s,z)}{\partial z^n}\bigg|_{z=0}, \tag{1}$$

$$F_k(s) = \sum_n n(n-1)\cdots(n-k+1)\,P_n(s) \;=\; \frac{\partial^k Q(s,z)}{\partial z^k}\bigg|_{z=1}, \qquad k = 1,2,\ldots \tag{2}$$

The mean multiplicity is $\langle N\rangle = F_1 =: \bar{n}$ and the second factorial moment is $F_2 = \langle N(N-1)\rangle$. Using these, the variance of the multiplicity distribution is

$$\begin{aligned}\mathrm{Var}(N) = \sigma^2 = \langle N^2\rangle - \langle N\rangle^2 &= F_2 + \langle N\rangle - \langle N\rangle^2 \\ &= f_2 + \bar{n}, \end{aligned} \tag{3}$$

where we defined the second *correlative* moment $f_2 := F_2 - \bar{n}^2$.

Giovannini's analysis shows that, to first order in α_s, the Markovian character of the elementary branchings leads naturally to a stochastic description of quark and gluon jets. The generating functions for a gluon jet and a quark jet (including both gluon and quark multiplicities) can be written as two-variable probability generating functions:

$$G(u_g, u_q; Y) = \sum_{n_g, n_q \geq 0} P_{1,0;n_g,n_q}(Y)\, u_g^{n_g} u_q^{n_q}, \tag{4}$$

$$Q(u_g, u_q; Y) = \sum_{n_g, n_q \geq 0} P_{0,1;n_g,n_q}(Y)\, u_g^{n_g} u_q^{n_q}, \tag{5}$$

where $P_{m_g,m_q;n_g,n_q}(Y)$ denotes the probability that an initial ensemble of m_g gluons and m_q quarks evolves, across a jet of thickness Y, into n_g gluons and n_q quarks. The "thickness" Y is a natural QCD evolution parameter, for example one may parametrize it as

$$Y = \frac{1}{2\pi b} \ln\left[1 + \alpha_b \ln \frac{Q^2}{\mu^2}\right],$$

with $Y = 0$ at $Q = \mu$. From a probabilistic viewpoint the total initial population of $(m_g + m_q)$ partons behaves as a superposition of $(m_g + m_q)$ independent one–parton populations, each evolving according to the same branching law which this reflects the branching Markov-chain nature of the cascade and permits factorized solutions for the parton multiplicity distributions.

3 Particular solution cases(Gluon and Quark Jet)

For Gluon Jet the distribution is calculated the probability distribution (starts at $n_g = 1$) where n_g is number of gluons

$$P_{n_g} = e^{-AY}\left(1 - e^{-AY}\right)^{n_g - 1}, \qquad n_g \geq 1.$$

The generating function is

$$G(z) = \sum_{n_g=1}^{\infty} P_{n_g} z^{n_g} = \sum_{n_g=1}^{\infty} e^{-AY}\left(1 - e^{-AY}\right)^{n_g-1} z^{n_g} \tag{6}$$

$$= e^{-AY} z \sum_{n_g=1}^{\infty} \left[(1 - e^{-AY})z\right]^{n_g-1} = \frac{e^{-AY} z}{1 - (1 - e^{-AY})z} = e^{-AY} z\left[1 - (1 - e^{-AY})z\right]^{-1}. \tag{7}$$

Differentiate $G(z)$ with respect to z:

$$G'(z) = \frac{d}{dz}\left(\frac{e^{-AY} z}{1 - (1 - e^{-AY})z}\right) = e^{-AY}\left[1 - (1 - e^{-AY})z\right]^{-2}, \tag{8}$$

$$G''(z) = \frac{d}{dz} G'(z) = 2(1 - e^{-AY})\, e^{-AY}\left[1 - (1 - e^{-AY})z\right]^{-3}. \tag{9}$$

Moments (evaluate derivatives at $z = 1$):

$$\langle n_g \rangle = G'(1) = \frac{e^{-AY}}{\big(1-(1-e^{-AY})\big)^2} = \frac{e^{-AY}}{(e^{-AY})^2} = e^{AY}.$$

$$\langle n_g(n_g-1) \rangle = G''(1) = 2(1-e^{-AY})\, e^{-AY} \big(e^{-AY}\big)^{-3} = 2(1-e^{-AY})\, e^{2AY} = 2e^{2AY} - 2e^{AY}.$$

From these we can also get $\langle n_g^2 \rangle$ and the variance:

$$\langle n_g^2 \rangle = \langle n_g(n_g-1) \rangle + \langle n_g \rangle = (2e^{2AY} - 2e^{AY}) + e^{AY} = 2e^{2AY} - e^{AY},$$

$$\mathrm{Var}(n_g) = \langle n_g^2 \rangle - \langle n_g \rangle^2 = (2e^{2AY} - e^{AY}) - e^{2AY} = e^{2AY} - e^{AY} = e^{AY}(e^{AY} - 1).$$

The second factorial cumulant (often denoted f_2) is

$$f_2 \equiv \langle n_g(n_g-1) \rangle - \langle n_g \rangle^2 = (2e^{2AY} - 2e^{AY}) - e^{2AY} = e^{2AY} - 2e^{AY} = e^{AY}\big(e^{AY} - 2\big).$$

Hence $f_2 > 0$ when $e^{AY} > 2$, indicating a wider distribution in that regime.

Similarly, for quark jet we have

$$P(n_g; Y) = \frac{\mu(\mu+1)\cdots(\mu+n_g-1)}{n_g!}\, e^{-\tilde{A}Y}\big(1-e^{-AY}\big)^{n_g}, \qquad \mu = \frac{\tilde{A}}{A},$$

which is a Negative-Binomial (Pascal) distribution. Introduce

$$p \equiv e^{-AY}, \qquad 1-p = 1-e^{-AY}.$$

Using $\mu = \tilde{A}/A$ we have $e^{-\tilde{A}Y} = (e^{-AY})^{\mu} = p^{\mu}$, so the PMF can be written as

$$P(n_g) = \binom{\mu+n_g-1}{n_g} p^{\mu}(1-p)^{n_g}, \qquad n_g = 0, 1, 2, \ldots$$

This is the negative-binomial distribution with parameters $r = \mu$ and success-probability p.

Probability generating function (PGF).

$$G(z) \equiv \sum_{n_g=0}^{\infty} P(n_g) z^{n_g} = p^{\mu} \sum_{n=0}^{\infty} \binom{\mu+n-1}{n} \big((1-p)z\big)^n = p^{\mu}\big(1-(1-p)z\big)^{-\mu}.$$

First derivative of the PGF. Differentiate $G(z)$ with respect to z:

$$G'(z) = \frac{d}{dz}\left[p^{\mu}\big(1-(1-p)z\big)^{-\mu}\right] = \mu\, p^{\mu}\big(1-(1-p)z\big)^{-\mu-1}(1-p).$$

Evaluate at $z = 1$ to obtain the mean (first factorial moment):

$$\langle n_g \rangle = G'(1) = \mu\, p^{\mu}\big(1-(1-p)\big)^{-\mu-1}(1-p) = \mu\, p^{\mu} p^{-\mu-1}(1-p) = \mu \frac{1-p}{p}.$$

Returning to $p = e^{-AY}$,

$$\boxed{\langle n_g \rangle = \mu\big(e^{AY} - 1\big).}$$

Second derivative of the PGF (gives the second factorial moment). Differentiate $G'(z)$ once more:

$$G''(z) = \frac{d}{dz}\left[\mu\, p^{\mu}(1-(1-p)z)^{-\mu-1}(1-p)\right] = \mu(\mu+1)p^{\mu}(1-(1-p)z)^{-\mu-2}(1-p)^2.$$

Evaluate at $z = 1$:

$$\langle n_g(n_g-1)\rangle = G''(1) = \mu(\mu+1)p^{\mu}p^{-\mu-2}(1-p)^2 = \mu(\mu+1)\frac{(1-p)^2}{p^2}.$$

Replacing $p = e^{-AY}$ yields

$$\boxed{\langle n_g(n_g-1)\rangle = \mu(\mu+1)\big(e^{AY}-1\big)^2.}$$

Useful derived quantities.

- **Normalized second factorial moment** (often denoted F_2):

$$F_2 \equiv \frac{\langle n_g(n_g-1)\rangle}{\langle n_g\rangle^2} = \frac{\mu(\mu+1)(e^{AY}-1)^2}{\mu^2(e^{AY}-1)^2} = \frac{\mu+1}{\mu} = 1+\frac{1}{\mu}.$$

  Note: F_2 is independent of Y.

- **Second cumulant (correlation)**

$$\langle n_g(n_g-1)\rangle - \langle n_g\rangle^2 = \mu(\mu+1)X^2 - \mu^2X^2 = \mu X^2, \qquad X \equiv e^{AY}-1,$$

  so

$$\boxed{\langle n_g(n_g-1)\rangle - \langle n_g\rangle^2 = \mu\big(e^{AY}-1\big)^2.}$$

- **Variance.** Use $\mathrm{Var}(n_g) = \langle n_g(n_g-1)\rangle + \langle n_g\rangle - \langle n_g\rangle^2$:

$$\begin{aligned}\mathrm{Var}(n_g) &= \mu(\mu+1)X^2 + \mu X - \mu^2X^2 = \mu X(1+(\mu+1)X-\mu X) \quad \text{(simplify)}\\ &= \mu X(1+X) = \mu(e^{AY}-1)e^{AY},\end{aligned}$$

  therefore

$$\boxed{\mathrm{Var}(n_g) = \mu\big(e^{2AY}-e^{AY}\big).}$$

  (Equivalently $\mathrm{Var}(n_g) = \langle n_g\rangle + \dfrac{\langle n_g\rangle^2}{\mu}$.)

Now we see for both gluon and quark jet second correlation is positive so gluons will be producing in bunch that is jet for baseline possion distribution it would be independently produced.

4 Hadronization

This is the second stage of the two–stage model where quarks and gluons convert into stable hadrons.

We consider the process $e^+e^- \to$ hadrons and introduce the following parameters:

$$\mu = k_p, \quad N,\ \bar{n}^h,\ \alpha,\ \bar{m}.$$

Definitions:

- **N**: Maximum number of hadrons that can be created or formed by a quark during the hadronisation stage.
- $\bar{\boldsymbol{n}}^{\boldsymbol{h}}$: Average number of hadrons under the same condition.
- $\boldsymbol{\alpha} = \dfrac{\bar{n}_g}{\bar{n}_q} = \dfrac{N_g}{N_q}$
- $\bar{\boldsymbol{m}}$: Average multiplicity value of gluons formed at the first stage.

The probability P_n of producing n hadrons is given by

$$P_n \ = \ \sum_{m=0}^{\infty} \frac{(k_p + m - 1)!}{m!\,(k_p - 1)!} \left(\frac{k_p}{k_p + \bar{m}}\right)^{k_p} \left(\frac{\bar{m}}{k_p + \bar{m}}\right)^{m}.$$

with normalisation
The generating-function representation:

$$P_n = \frac{1}{n!}\,\frac{\partial^n}{\partial z^n}\left(1 + \frac{\bar{n}^h}{N} z\right)^{(2+\alpha)mN}\Bigg|_{z=0} = \frac{1}{n!}\,\frac{\partial^n Q}{\partial z^n}\Bigg|_{z=0}.$$

A combined two-stage expression:

$$P_n \ = \ \Omega \sum_{m=0}^{\infty} \frac{(k_p + m - 1)!}{m!\,(k_p - 1)!} \left(\frac{k_p}{k_p + \bar{m}}\right)^{k_p} \left(\frac{\bar{m}}{k_p + \bar{m}}\right)^{m} \times \binom{(2+\alpha)mN}{n} \left(\frac{\bar{n}^h}{N}\right)^{n} \left(1 - \frac{\bar{n}^h}{N}\right)^{(2+\alpha)mN - n}.$$

For Bernoulli distribution (each source produces at most one hadron with probability q):

$$P(n \mid m) = \binom{m}{n} q^n (1-q)^{m-n}, \qquad n = 0, 1, \ldots, m.$$

If the parton-stage multiplicity distribution is $P_m^{(p)}$ with mean $\bar{m} = \sum_{m=0}^{\infty} m\, P_m^{(p)}$, the final hadron multiplicity (is possibly restricted to even values via Ω) is

$$P_n \ = \ \sum_{m=n}^{\infty} P_m^{(p)} \binom{m}{n} q^n (1-q)^{m-n}.$$

The generating function form is

$$G(z) = \sum_{n=0}^{\infty} P_n z^n = \sum_{m=0}^{\infty} P_m^{(p)} \big(1 - q + qz\big)^m.$$

The first two factorial moments follow from derivatives at $z = 1$:

$$\langle N \rangle = G'(1) = q \sum_{m=0}^{\infty} m\, P_m^{(p)} \;=\; q\, \bar{m},$$

$$\langle N(N-1) \rangle = G''(1) = q^2 \sum_{m=0}^{\infty} m(m-1)\, P_m^{(p)}.$$

The second correlation (second factorial cumulant) f_2 is

$$\begin{aligned} f_2(N) &\equiv \langle N(N-1) \rangle - \langle N \rangle^2 \\ &= q^2 \sum_{m=0}^{\infty} m(m-1) P_m^{(p)} - \big(q \sum_{m=0}^{\infty} m P_m^{(p)}\big)^2 \\ &= q^2 \left(\sum_{m=0}^{\infty} m(m-1) P_m^{(p)} - \left(\sum_{m=0}^{\infty} m P_m^{(p)} \right)^2 \right). \end{aligned}$$

Using $\mathrm{Var}(M) = \sum m^2 P_m^{(p)} - \bar{m}^2$ and $\sum m(m-1) P_m^{(p)} = \mathrm{Var}(M) + \bar{m}^2 - \bar{m}$, the above can be rewritten compactly as

$$\boxed{f_2(N) = q^2 \big(\mathrm{Var}(M) - \bar{m}\big)}.$$

If you enforce even-only hadron multiplicities for e^+e^- (i.e. set $P_n = 0$ for odd n) apply a renormalization factor Ω to even-n probabilities and recompute the sums above; the boxed formula assumes the simple even-weighting Ω shown explicitly.This give the negative second correlation moment which agree with experimental data.
In most cases we take the normalization constant as:

$$\Omega = 2.$$

5 Fitting of Experimental Data

Here are the plots of all four fitted experimental energies.

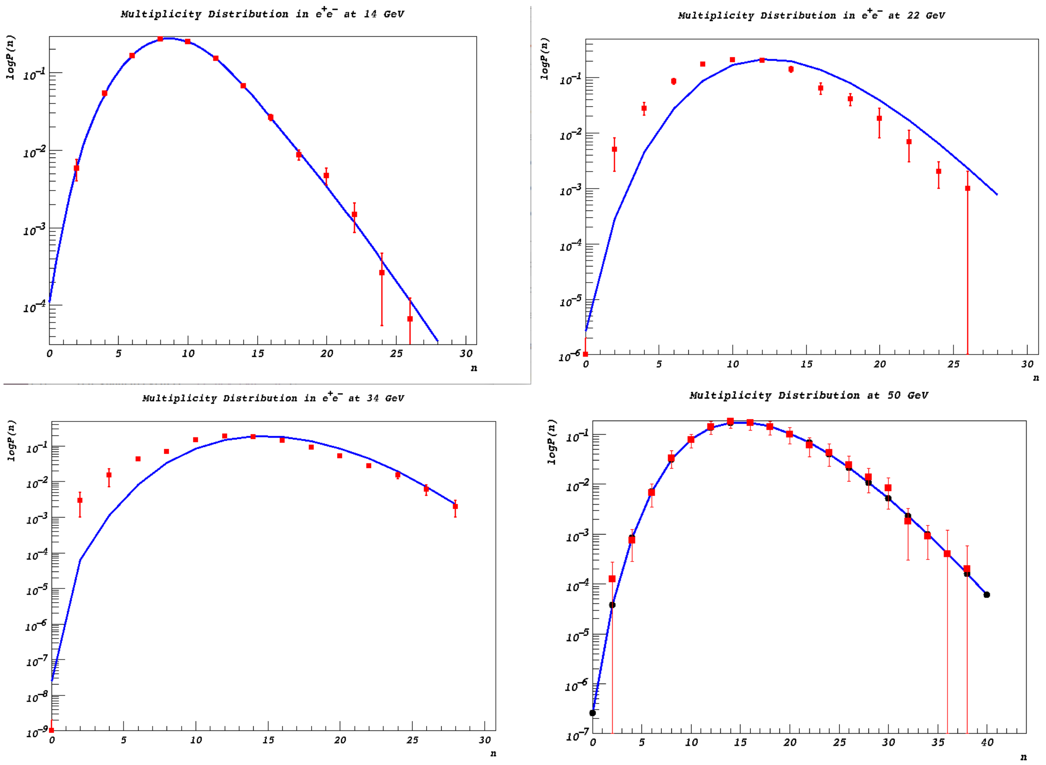

Figure 1: Fitted multiplicity distributions for 14, 22, 34 and 50 GeV.

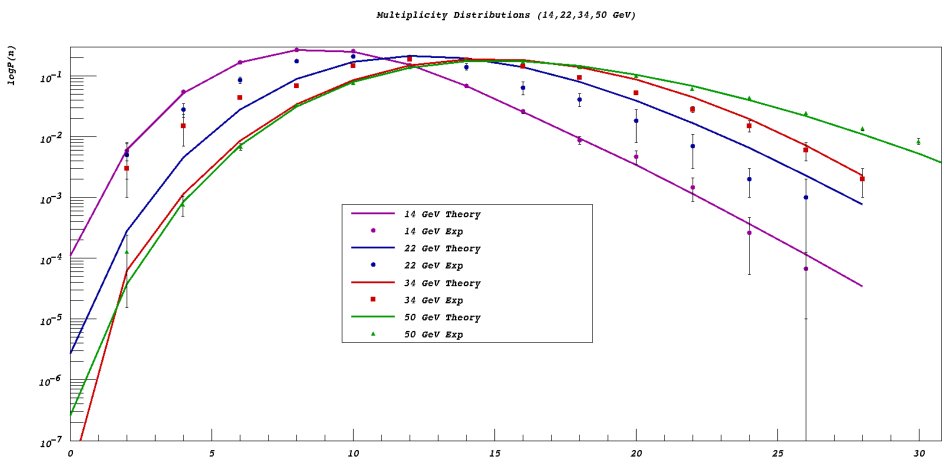

Figure 2: Comparison of fits across all energies.

6 References

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3.*Charged-particle multiplicities in e^+e^- annihilations at $\sqrt{s}$ = 50–61.4 GeV*.

4.CERN-ROOT Software for fitting Data.