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FINAL REPORT ON THE INTEREST PROGRAMME

Puzzles of Multiplicity

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Abstract

The Giovannini approach was aplied to describe the e^+e- annihilation process trough a two stage model TSM. The theoretical model was compared with experimental data from PETRA, computating the expressions in C++ with the support of CERN Root libraries for minimizing and fitting. The results showed that TSM has a good correspondence with experimental date, so it is able to describe the process of quark-gluon cascades cand its convertion to hadrons.

Introduction

Multiparticle production (MP) processes are fundamental in the study of high energy physics. Thus, with an increase of the energy of the accelerators new channels of reactions were opened, in which the number of secondaries increased and new particles were born. In consequence phenomenological models and new theories were developed, such as quantum chromodynamics (QCD) for explaining strong interactions.

One of the observables in the experiment is the multiplicity - the number of secondary particles, which can be charged or neutral. The most predominant among them is the multiplicity of charged particles. To understand and describe experiments carried out at accelerators, Monte Carlo generators are built along with QCD description of quark-gluon interactions. However, these generators have significant deviations when they try to fit with the data from high energy hadron experiments; hence, a two stage model (TSM) is introduced to explain the quark-gluon cascade through perturbative QCD and then the hadronization process by phenomenological models built empirically from experience.

One of the most common examples for MP is in e^+e^- annihilation at high energy, where more than 60 secondary particles can be produced. In accordance with QCD, the annihilation may occur through the creation of a γ or Z^0 into two quarks:

$$e^+e^- \to (\gamma, Z^0) \to \bar{q}q$$

The first stage, takes place at high energy when the strong coupling α_s is small, so the process of partons fission can be described with perturbative QCD. After it, in the second stage, the quarks and gluons have lost enough energy that they start to transform into observable hadrons. At the end, large amount of data from secondary particles in MP is collected experimentally, therefore the analysis of the whole process should be carry out using statistical methods.

Even though certain features of the predictions at the parton level are expected to be insensitive to details of the hadronization mechanism; they were tested directly in [1] by using hadron distributions in a range of energies between 50-61.4 and 172-189 GeV. This study showed that TSM fits good with the experimental data about e^+e^-, no contradictions appeared.

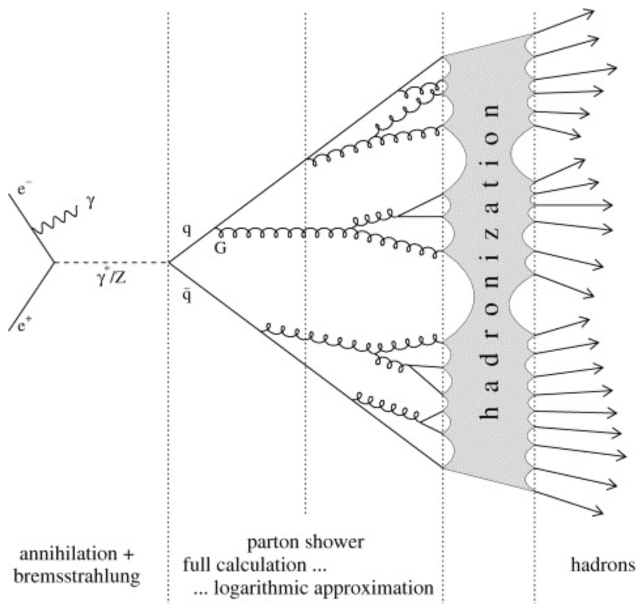

Figure 1: Scheme of the TSM process in e^+e^- annihilation.

During the 6 weeks of the INTEREST project, we will work in the detailed explanation, calculation and comprehension of the expresions from the statistical distributions for e^+e^- annihilation according to the TSM. Besides, a program in C++ should be written to do the calculations of multiplicity for the energies 14 and 22 GeV through statistical formulas; and using the CERN root libraries for fitting, the results will be fit and compare with the experimental data from the Table 3 in [2].

1 First Stage: Cascade

To study the MP processes at high energy, we based on the approach of Giovanini [3] to describe quark and gluon jets as Markov branching processes. He recovered the results of Konishi, Ukawa and Veneciano, working at the leading logarithm approximation avoiding IR divergences by considering finite x, stablishing this way the probabilistic nature of the problem. To describe how partons branch as they go from a high energy scale down to lower energies, we use the QCD parameter Y. It measures the time evolution of a parton shower.

$$Y = \frac{1}{2\pi b} log[1 + (\alpha b) log \frac{Q^2}{\mu^2}] \tag{1}$$

where $2\pi b = \frac{1}{6}(11N_c - 2N_f)$ for a theory with N_c colours and N_f flavours, as the thickness of the jets and their development as Markov process.

There are three elementary processes that take place in partons at high energy that contribute to the formation of QCD jets. Each one develops independently from the others and with the same infinitesimal probability. Assuming three Y-independent parameters $A, \tilde{A}$ and B we can define its probabilities as follows:

1. **Quark Bremsstrahlung** $q \to q + g$
   With $\tilde{A}\Delta Y$ the probability that a quark radiate a gluon and then continue on its way.

2. **Gluon Fission** $g \to g + g$
   With $A\Delta Y$ the probability that a gluon in the infinitesimal interval ΔY will convert into two gluons.

3. **Quark Pair Creation** $g \to \bar{q} + q$
   With $B\Delta Y$ the probability that a quark-antiquark pair will be created from a gluon.

The infinitesimal proabiblity for a gluon or a quark to convert into m_g gluons and m_q quarks in the interval $(Y, Y + \Delta Y)$ is given by the sum of the next probabilities for a gluon (2) and for a quark (3) respectively:

$$\delta_{1,m_g}\delta_{0,m_q} + a^{(g)}_{m_g,m_q}\Delta Y + o(\Delta Y) \tag{2}$$

$$\delta_{0,m_g}\delta_{1,m_q} + a^{(q)}_{m_g,m_q}\Delta Y + o(\Delta Y) \tag{3}$$

due to only the three fundamental processes are allowed, we obtain the probability (considering its conservation and its normalization) for gluon (4) and for quark (5):

$$1 + a^{(g)}_{1,0}\Delta Y + a^{(g)}_{2,0}\Delta Y + a^{(g)}_{0,2}\Delta Y + o(\Delta Y) = 1 \tag{4}$$

notice that $a^{(g)}_{1,0} + a^{(g)}_{2,0} + a^{(g)}_{0,2} = 0$, implies that $a^{(g)}_{1,0} = -(a^{(g)}_{2,0} + a^{(g)}_{0,2}) < 0$.

$$1 + a^{(q)}_{0,1}\Delta Y + a^{(q)}_{1,1}\Delta Y + o(\Delta Y) = 1 \tag{5}$$

in this case $a^{(q)}_{0,1} + a^{(q)}_{1,1} = 0$, so $a^{(q)}_{0,1} = -a^{(q)}_{1,1} < 0$.
Additionally, we can realize that is convinient to define:
$a^{(g)}_{2,0} = A$ for gluon fission.
$a^{(g)}_{0,2} = B$ for pair creation.
$a^{(q)}_{1,1} = \tilde{A}$ for Bremsstrahlung raditation.

Using two auxiliary variables u_g and u_q we define the infinitesimal generating functions for gluon and quark respectively:

$$w^{(g)}(u_g, u_q) = \sum_{m_g,m_q=0}^{\infty} a^{(g)}_{m_g,m_q} u_g^{m_g} u_q^{mq} = (-A - B)u_g + Au_g^2 + Bu_q^2 \tag{6}$$

$$w^{(q)}(u_g, u_q) = \sum_{m_g,m_q=0}^{\infty} a^{(q)}_{m_g,m_q} u_g^{m_g} u_q^{mq} = -\tilde{A}u_q + \tilde{A}u_q u_g \tag{7}$$

Now we introduce the probability generaction functions corresponding to gluon and quark jets:

$$G(u_g, u_q; Y) = \sum_{n_g,n_q=0}^{\infty} P_{1,0;n_g,n_q}(Y) u_g^{n_g} u_q^{n_q} \tag{8}$$

$$Q(u_g, u_q; Y) = \sum_{n_g,n_q=0}^{\infty} P_{0,1;n_g,n_q}(Y) u_g^{n_g} u_q^{n_q} \tag{9}$$

Considering the fact that partons evolute individually, it leads us to:

$$\sum_{n_g,n_q=0}^{\infty} P_{m_g,m_q;n_g,n_q}(Y) u_g^{n_g} u_q^{n_q} = [G(u_g, u_q; Y)]^{m_g} [Q(u_g, u_q; Y)]^{m_q} \tag{10}$$

Moreover, since the process is homogenous in Y the transition probabilities obey Chapman-Kolmogorov equations:

$$P_{m_g,m_q;n_g,n_q}(Y + Y') = \sum_{l_g,l_q=0}^{\infty} P_{m_g,m_q;l_g,l_q}(Y) P_{l_g,l_q;n_g,n_q}(Y') \tag{11}$$

So the generating fucntions (8) and (9) became into:

$$G(u_g, u_q; Y + Y') = G[G(u_g, u_q; Y'), Q(u_g, u_q; Y'); Y] \tag{12}$$

$$Q(u_g, u_q; Y + Y') = Q[G(u_g, u_q; Y'), Q(u_g, u_q; Y'); Y] \tag{13}$$

Substituting Y' for an infinitisemal ΔY and writting in terms of the infinitesimal generating functions the previous expressions now are:

$$G(u_g, u_q; Y + \Delta Y) = G[u_g + w^{(g)}(u_g, u_q)\Delta Y + o(\Delta Y), u_q + w^{(q)}(u_g, u_q)\Delta y + o(\Delta Y); Y] \tag{14}$$

$$Q(u_g, u_q; Y + \Delta Y) = Q[u_g + w^{(g)}(u_g, u_q)\Delta Y + o(\Delta Y), u_q + w^{(q)}(u_g, u_q)\Delta y + o(\Delta Y); Y] \tag{15}$$

We can make a Taylor serie to the variables $u_g + w^{(g)}(u_g, u_q)\Delta Y + o(\Delta Y)$ and on $u_q + w^{(q)}(u_g, u_q)\Delta Y + o(\Delta Y); andaftertakedelimitwhen \Delta Y \to 0$.

$$\frac{\partial G(u_g, u_q; Y)}{\partial Y} = \frac{\partial G}{\partial u_g} w^{(g)}(u_g, u_q) + \frac{\partial G}{\partial u_q} w^{(q)}(u_g, u_q) \tag{16}$$

$$\frac{\partial Q(u_g, u_q; Y)}{\partial Y} = \frac{\partial Q}{\partial u_g} w^{(g)}(u_g, u_q) + \frac{\partial Q}{\partial u_q} w^{(q)}(u_g, u_q) \tag{17}$$

The previous two equations correspond to forward Komlmogorov equations for the generation functions of the transition probabilities $P_{m_g,m_g;n_g,n_q}(Y)$. However, to solve completely our problem we need the corresponding backward Kolmogorov equations:

$$\frac{\partial G}{\partial Y} = w^{(g)}[G(u_g, u_q; Y), Q(u_g, u_q; Y)] \tag{18}$$

$$\frac{\partial Q}{\partial Y} = w^{(q)}[G(u_g, u_q; Y), Q(u_g, u_q; Y)] \tag{19}$$

insertig the initial conditions in both cases $G(u_g, u_q; 0) = u_g$ and $Q(u_g, u_q; 0) = u_q$

Taking in consideration that only the three elementary particle processes are allowed, we get:

$$\frac{\partial G}{\partial Y} = -AG + AG^2 - BG + BQ^2 \tag{20}$$

$$\frac{\partial Q}{\partial Y} = -\tilde{A}Q + \tilde{A}QG \tag{21}$$

In addition, we need to define the following probability parameters:

$B \equiv \frac{1}{3}N_c$; $\tilde{A} \equiv \frac{1}{\epsilon}[\frac{N_c^2-1}{2N_c}] = A_0^{qg}$; $A \equiv \frac{N_c}{\epsilon} = A_0^{gg}$

for the following calculations we will neglect B, we will consider it as $B = 0$ so no quark air creation will happen. Thus, we get the following differential equations for jets:

$$\frac{dP_{1,0;n_g,0}(Y)}{dY} = A(n_g-1)P_{1,0;n_g-1,0}(Y) - An_gP_{1,0;n_g,0}(Y) \tag{22}$$

$$\frac{dP_{0,1;n_g,1}(Y)}{dY} = -\tilde{A}P_{0,1;n_g,1}(Y) - An_gP_{0,1;n_g,1}(Y) + \tilde{A}P_{0,1;n_g-1,1}(Y) + A(n_g-1)P_{0,1;n_g-1,1}(Y) \tag{23}$$

using the following initial conditions to solve:

$P_{1,0;1,0}(0) = 1$; $P_{1,0;n_g,0}(0) = 0$; $\forall n_g > 1$

$P_{0,1;0,1}(0) = 1$; $P_{0,1;n_g,1}(0) = 0$; $\forall n_g \geq 1$

Then, the solutions we found for each parton are:

- Gluon jet

$$P_{1,0;1,0}(Y) = e^{-AY} \tag{24}$$

$$P_{1,0;n_g,0}(Y) = e^{-AY}(1-e^{-AY})^{n_g-1} \tag{25}$$

$$\langle n_g \rangle = e^{AY} \tag{26}$$

$$P_{1,0;n_g,0}(Y) = \frac{1}{\langle n_g \rangle}(1-\frac{1}{\langle n_g \rangle})^{n_g-1} \tag{27}$$

$$D_2 = \langle n_g^2 \rangle - \langle n_g \rangle^2 = e^{AY}(e^{AY} - 1) \tag{28}$$

$$\bar{f}_2 = e^{AY}(e^{AY} - 2) \tag{29}$$

Putting all together, we build the generating function definening the new parameter $\mu = \frac{\tilde{A}}{A}$:

$$G = \sum_{n_g=0}^{\infty} u_g^{n_g} P_{1,0;n_g,0}(Y) = \frac{u_g e^{-AY}}{1 - u_g(1 - e^{-AY})} \tag{30}$$

- Quark jet

$$P_{0,1;0,1}(Y) = e^{-\tilde{A}Y} \tag{31}$$

$$P_{0,1;n_g,1}(Y) = \frac{\mu(\mu+1)...(\mu+n_g-1)}{n_g!} e^{-\tilde{A}Y}(1 - e^{-AY})^{n_g} \tag{32}$$

$$\langle n_g \rangle = \mu(e^{AY} - 1) \tag{33}$$

$$P_{0,1;n_g,1}(Y) = \frac{\mu(\mu+1)...(\mu+n_g-1)}{n_g!} [\frac{\langle n_g \rangle}{\langle n_g \rangle + \mu}]^{n_g} [\frac{\mu}{\langle n_g \rangle + \mu}]^{\mu} \tag{34}$$

$$D_2 = \mu e^{AY}(e^{AY} - 1) \tag{35}$$

$$\bar{f}_2 = \mu(e^{AY} - 1)^2 \tag{36}$$

Finally this leads to:

$$Q = \sum_{n_g=0}^{\infty} u_g^{n_g} u_q P_{0,1;n_g,1}(Y) = u_q[\frac{e^{-AY}}{1 - u_g(1 - e^{-AY})}]^{\mu} \tag{37}$$

2 Second Stage: Hadronization

For this step, we will require to put together all the previous calculations we have done. We add stage of hadronization to parton stage with aid of its factorization. Supposing that hypothesis of soft colourless is right, we can write the MD as follows:

$$P_n(s) = \sum_{m=0}^{\infty} P_m^P P_m^H(m, s) \tag{38}$$

Remember that the MD for hard fissionis described by Negative Binomial distribution NBD for a quark jet, so redefining parameters $k_p = \frac{\tilde{A}}{A}$, $\bar{m} = \sum_m m P_m^P$ the expression obtained:

$$P_m^P(s) = \frac{k_p(k_p+1)...(k_p+m-1)}{m!}(\frac{\bar{m}}{\bar{m}+k_p})^m(\frac{k_p}{\bar{m}+k_p})^{k_p} \tag{39}$$

With its associated generating function:

$$Q_m^P(s,z) = [1+\frac{\bar{m}}{k_p}(1-z)]^{k_p} \tag{40}$$

On the other hand, for the process of hadrons formed from m partons we can write the MD as:

$$P_n^H = C_{k_p}^n(\frac{\bar{n}_p^h}{N_p})^n(1-\frac{\bar{n}_p^h}{N_p}(z-1))^{N_p-n} \tag{41}$$

and its generating function:

$$Q_p^H = [1+\frac{\bar{n}_p^h}{N_p}(z-1)]^{N_p} \tag{42}$$

where N_p is the maximun numbre of secondary particles, $\bar{n}_p^h$ the average multiplicity. If we introduce a new consideration, that probability for producing a parton either from a quark or from a gluon is the same defining the ratio $\alpha = \frac{N_g}{N_q}$. Making additional changes to simplify notation $N = N_q$, $\bar{n}_q^h = \bar{n}^h$ the generating functions in (40) and (42) take the form:

$$Q_q^H = (1+\frac{\bar{n}^h}{N}(z-1))^N \tag{43}$$

$$Q_g^H = (1+\frac{\bar{n}^h}{N}(z-1))^{\alpha N} \tag{44}$$

Doing the combination of the two stages for the e^+e^- annihilation, expression (38) turns into:

$$P_n(s) = \sum_{m=0}^{\infty} P_m^P \frac{\partial^n}{\partial z^n}(Q^H)^{2\alpha+m}|_{z=0} \tag{45}$$

Finally obtaning the multiplicity distribution:

$$P_n = \sum_{m=0}^{\infty} P_m \frac{(2+\alpha m)N((2+\alpha m)N-1)...((2-\alpha m)N-n+1)}{n!}(\frac{\bar{n}^h}{N})^n(1-\frac{\bar{n}^h}{N})^{(2+\alpha m)(N-n)} \tag{46}$$

having two cases for P_m:

$P_m = (\frac{k_p}{k_p+\bar{m}})^{k_p}$; when m is =0

$$P_m = (\frac{k_p}{k_p+\bar{m}})^{k_p}(\frac{\bar{m}}{k_p+\bar{m}})^m \frac{k_p(k_p+1)..(k_p+m-1)}{m!}; \text{ when } m \neq 0$$

3 Calculations and fitting for e^+e^- annihilation

A code was programmed in C++ for computing the expresion (46) to obtain the multiplicity distributions for different values of energy. After, using the CERN Root libraries for minimazing (Minuit2) and for fitting, a comparison was made with experimental data from an article [2]. The results are shown in the following Table:

[GeV]	k_p	$\bar{m}$	$\bar{n}^h$	N	α	Ω	χ^2
14	19.999 ±8.631	0.083 ±0.327	4.465 ±.099	27.738 ±13.886	0.967 ±0.247	1.997 ±0.034	2.791
22	10.001 ±5.068	0.891887 ±0.579	4.847 ±0.262	27.998 ± 11.367	0.374 ±0.123	1.998 ±0.035	1.309

Table 1: Parameters calculated after fitting TSM on experimental data.

4 Conclusions

From the values obtained in the table above, we notice that the TSM is fitting with the observations on the experiments. Thus, this approach can describes satisfactory MP process at high energy, in articular, the MD for e^+e^- annihilation.

5 Acknowledgments

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