JOINT INSTITUTE FOR NUCLEAR RESEARCH
Bogoliubov laboratory of Theoretical Physics

FINAL REPORT ON THE INTEREST PROGRAMME

Numerical Methods in the Theory of Topological Solitons

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1 Introduction

Solitons are stable, localized field configurations that behave much like particles. They emerge in diverse nonlinear systems, spanning fields from nonlinear optics and condensed matter physics to nuclear physics and cosmology. Their existence in nonlinear theories is crucial because the standard linear models are often too simplistic to describe real-world physical situations. Unlike linear systems where phenomena are often treated perturbatively, solitons represent non-dissipative, non-perturbative features that are missed entirely by linear approximations.

A prime example is the kink, a one-dimensional, spatially localized soliton. Kinks appear in models where the potential has degenerate vacua, interpolating between them. Their stability can be topological (as in the sine-Gordon model) or depend on a precise dynamic balance between nonlinearity and dispersion (as seen later in the KdV equation). The search for solutions to nonlinear systems, and thus the existence of solitons, is closely linked to integrability. An integrable system, like the sine-Gordon model, possesses an infinite tower of conserved quantities, allowing for analytical solutions.

The $\phi^{\mathbf{4}}$ and $\phi^{\mathbf{6}}$ models, featuring polynomial potentials with double

or triple (ϕ^6) degenerate vacua, provide examples of non-integrable theories While ϕ^4 supports kinks, its non-integrability means soliton scattering is inelastic and often exhibits complex, chaotic dynamics not seen in integrable counterparts.

2 Sine Gordon Model

The sine-Gordon model is a simple, relativistic, nonlinear scalar field theory formulated by Perring and Skyrme in 1962.The Lagrangian for the model is given by :

$$L = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - U[\phi] \equiv \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - U(\phi). \tag{1}$$

where U(ϕ) is given by U(ϕ) = (1 - cosϕ) for the Sine Gordon model.
The Canonical stress-energy tensor is

$$T_{\mu\nu} = \left(\frac{\delta L}{\delta(\partial^\mu\phi)}\right)\partial_\nu\phi - g_{\mu\nu}L = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}L. \tag{1.12}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi), \qquad V(\phi) = 1 - \cos\phi.$$

3 Kinks/Antikinks in the Sine Gordon Model

Equation of Motion

The Euler–Lagrange equation is

$$\partial_\mu \partial^\mu \phi + \frac{dV}{d\phi} = 0.$$

For static solutions $\phi = \phi(x)$:

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi} = \sin\phi.$$

First Integral (Energy Conservation)

Multiply by $\frac{d\phi}{dx}$ and integrate once:

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 - (1 - \cos\phi) = C.$$

For finite-energy kink solutions connecting neighboring vacua we choose the integration constant $C = 0$. Hence

$$\frac{1}{2}\phi_x^2 = 1 - \cos\phi = 2\sin^2\frac{\phi}{2}.$$

Therefore

$$\frac{d\phi}{dx} = \pm 2\sin\frac{\phi}{2}.$$

Integration and Kink Solution

Separate variables (take the + sign for the kink that increases from 0 to 2π):

$$\frac{d\phi}{2\sin(\phi/2)} = dx.$$

Let $u = \phi/2$. Then $d\phi = 2\,du$ and the integral becomes

$$\int \frac{2\,du}{2\sin u} = \int \frac{du}{\sin u} = \int \csc u\,du = \ln\left|\tan\frac{u}{2}\right| + \text{const.}$$

Back-substituting and exponentiating yields the standard form

$$\ln\tan\frac{\phi}{4} = x - x_0 \quad \Rightarrow \quad \boxed{\phi_K(x) = 4\arctan\left(e^{x-x_0}\right)}.$$

The antikink (decreasing from 2π to 0) is obtained by the negative sign:

$$\boxed{\phi_{\bar{K}}(x) = 4\arctan\left(e^{-(x-x_0)}\right) = 4\arctan\left(e^{x_0-x}\right)}.$$

Vacua and Topology

The potential minima are at

$$\phi = 2\pi n, \qquad n \in \mathbb{Z},$$

so kinks interpolate between neighboring vacua (e.g. $0 \to 2\pi$). These are topological solitons with integer winding number.

Kink Energy (Mass)

The energy functional for static fields is

$$E[\phi] = \int_{-\infty}^{\infty} \left(\frac{1}{2}\phi_x^2 + V(\phi) \right) dx.$$

Using the first integral $\frac{1}{2}\phi_x^2 = 1 - \cos\phi = 2\sin^2(\phi/2)$, the energy density becomes

$$\mathcal{E} = \phi_x^2/2 + V = 2\sin^2\frac{\phi}{2} + 2\sin^2\frac{\phi}{2} = 4\sin^2\frac{\phi}{2}.$$

Change variable to ϕ: $dx = d\phi/\phi_x = d\phi/(2\sin(\phi/2))$. Thus

$$E = \int_0^{2\pi} 4\sin^2\frac{\phi}{2} \cdot \frac{d\phi}{2\sin(\phi/2)} = \int_0^{2\pi} 2\sin\frac{\phi}{2}\, d\phi.$$

Let $u = \phi/2$ so $d\phi = 2\,du$, and the limits go from 0 to π:

$$E = \int_0^{\pi} 4\sin u\, du = 4\big[-\cos u\big]_0^{\pi} = 4(2) = 8.$$

Hence the kink mass is

$$\boxed{E_{\text{kink}} = 8.}$$

Small Oscillations

The curvature at the vacuum $\phi = 0$ is

$$V''(0) = \frac{d^2}{d\phi^2}(1 - \cos\phi)\bigg|_{\phi=0} = 1,$$

so small linear excitations about the vacuum have mass $m^2 = 1$.

4. Kinks/Antikinks in ϕ^6 model

Find the classical kink (antikink) solution of the $1+1$ dimensional ϕ^6 model with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi),$$

where the potential is

$$V(\phi) = \frac{1}{2}\phi^2(1-\phi^2)^2.$$

Equation of Motion

The Euler–Lagrange equation is

$$\partial_\mu \partial^\mu \phi + \frac{dV}{d\phi} = 0.$$

For static solutions $\phi = \phi(x)$:

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}.$$

Derivative of the Potential

We compute

$$V(\phi) = \frac{1}{2}\phi^2(1-\phi^2)^2.$$

First expand:

$$(1-\phi^2)^2 = 1 - 2\phi^2 + \phi^4,$$

$$V(\phi) = \frac{1}{2}(\phi^2 - 2\phi^4 + \phi^6).$$

Thus,

$$\frac{dV}{d\phi} = \frac{1}{2}(2\phi - 8\phi^3 + 6\phi^5) = \phi - 4\phi^3 + 3\phi^5.$$

So the equation of motion becomes

$$\frac{d^2\phi}{dx^2} = \phi - 4\phi^3 + 3\phi^5.$$

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Bogomolny Equation

The potential can be written as

$$V(\phi) = \frac{1}{2}\left[\phi(1-\phi^2)\right]^2.$$

Therefore the first-order Bogomolny equation is

$$\frac{d\phi}{dx} = \pm\phi(1-\phi^2).$$

Integration

We separate variables:

$$\frac{d\phi}{\phi(1-\phi^2)} = dx.$$

Use partial fractions:

$$\frac{1}{\phi(1-\phi)(1+\phi)} = \frac{1}{\phi} + \frac{1/2}{1-\phi} - \frac{1/2}{1+\phi}.$$

Thus,

$$\int \left(\frac{1}{\phi} + \frac{1/2}{1-\phi} - \frac{1/2}{1+\phi}\right) d\phi = x - x_0.$$

Integrating:

$$\ln|\phi| - \frac{1}{2}\ln|1-\phi| - \frac{1}{2}\ln|1+\phi| = x - x_0.$$

Combine logs:

$$\ln\left(\frac{\phi}{\sqrt{1-\phi^2}}\right) = x - x_0.$$

Exponentiating:

$$\frac{\phi}{\sqrt{1-\phi^2}} = e^{x-x_0}.$$

Squaring:

$$\frac{\phi^2}{1-\phi^2} = e^{2(x-x_0)}.$$

Solving:

$$\phi^2 = \frac{e^{2(x-x_0)}}{1+e^{2(x-x_0)}} = \frac{1}{1+e^{-2(x-x_0)}}.$$

Hence the kink solution is

$$\phi_K(x) = \sqrt{\frac{1}{1+e^{-2(x-x_0)}}} = \sqrt{\frac{1}{2}\left(1+\tanh(x-x_0)\right)}$$

The antikink is

$$\phi_{\bar{K}}(x) = \sqrt{\frac{1}{1+e^{2(x-x_0)}}} = \sqrt{\frac{1}{2}\left(1-\tanh(x-x_0)\right)}$$

Vacua and Topology

The vacua of the model follow from

$$V(\phi) = 0 \Rightarrow \phi = 0, \quad \phi = \pm 1.$$

Thus:

- The kink connects $\phi = 0 \to \phi = 1$
- The antikink connects $\phi = 1 \to \phi = 0$

They are topological solitons.
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Energy of the Kink

The energy is

$$E = \int_{-\infty}^{\infty} \left(\frac{1}{2}\phi_x^2 + V(\phi) \right) dx.$$

Using the Bogomolny equation $\phi_x^2 = 2V$:

$$E = \int_0^1 \phi(1-\phi^2) d\phi.$$

$$E = \left[\frac{1}{3}\phi^2 - \frac{1}{5}\phi^4 \right]_0^1 = \boxed{\frac{2}{15}}.$$

$$\phi_K(x) = \sqrt{\tfrac{1}{2}(1+\tanh(x-x_0))}, \quad \phi_{\bar{K}}(x) = \sqrt{\tfrac{1}{2}(1-\tanh(x-x_0))}$$

These are the classical kink and antikink solutions of the ϕ^6 model.

5. Interaction of solitons in the ϕ^6 model

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\phi^2(\phi^2-1)^2.$$

The force is

$$F = \left[-\frac{1}{2}\left((\partial_t\phi)^2 + (\partial_x\phi)^2\right) + U(\phi)\right]_{-\infty}^{r}.$$

Case 1

$$\phi(x) = \phi_{(1,0)}(x+d) + \phi_{(0,1)}(x-d).$$

$$\phi_{(0,1)}(x) = \sqrt{\frac{1+\tanh x}{2}}, \quad \phi_{(1,0)}(x) = \sqrt{\frac{1-\tanh x}{2}}.$$

$$F = \left[-\frac{1}{2}(\partial_x\phi)^2 + U(\phi)\right]_{-\infty}^{r}.$$

$$F(r) = 2e^{-2d} = 2e^{-R}, \quad R = 2d.$$

$$E_{\text{int}}(R) = -2e^{-R}.$$

Case 2

$$\phi(x) = \phi_{(0,1)}(x+d) + \phi_{(1,0)}(x-d) - 1.$$

$$F(r) = 2e^{-4d} = 2e^{-2R}.$$

$$E_{\text{int}}(R) = -e^{-2R}.$$

The negative sign confirms that the interaction between kink and antikink is attractive, and the exponential decay reflects the localized nature of the soliton tails.

6. Kinks/Antikinks in the deformed Sine-Gordon potential

We now find kinks/antikinks soltions using numerical methods in the deformed sine gordon potential which is a mixture of Sine-Gordon and Quartic Double-Well Polynomial potential. The Lagrangian is

$$L = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi),$$

where

$$V(\phi) = (1-\epsilon)(1-\cos\phi) + \frac{\epsilon\phi^2}{8\pi^2}(\phi - 2\pi)^2.$$

The numerical solutions along with the potential for $\epsilon = 0$ and 2.7 are shown.

•For small ϵ (near 0), potential = (1-cos ϕ) so standard sine-Gordon-like behavior: minima near integer multiples of 2π, kinks connecting adjacent minima ($0 \to 2\pi$).

• Increasing ϵ gradually deforms the potential; up to moderate ϵ the two minima near 0 and 2π persist and a $0 \to 2\pi$ kink exists.

• For large enough ϵ (here at $\epsilon = 2.7$) a qualitative change occurs: a local minimum near $\phi = \pi$ appears and even becomes energetically favorable . That means new kink types appear ($0 \leftrightarrow \pi$ and $\pi \leftrightarrow 2\pi$) and the physics of kinks changes.

• At $\epsilon = 2.7$ the large magnitude of the field-dependent term dominates, making the potential locally negative between the vacua, which suggests the 1+1 dim system is likely dynamically unstable or that the "kink" solution is highly distorted and meta-stable, potentially decaying into an unstable region if the field is perturbed.

7. Conclusion

In this report, we investigated the existence and properties of kink and antikink soliton solutions in several nonlinear scalar field theories, namely the sine-Gordon model, the ϕ^6 model, and a deformed sine-Gordon model that interpolates between a periodic and a polynomial potential. These models illustrate how nonlinearity and vacuum structure lead naturally to topologically stable, particle-like excitations.

Overall, this study demonstrates how the vacuum structure and degree of integrability govern the existence, stability, and interaction of solitons. The transition from integrable to non-integrable models leads to energy radiation, attractive forces between solitons, and metastability, which are essential features in realistic physical systems ranging from condensed matter to cosmology.

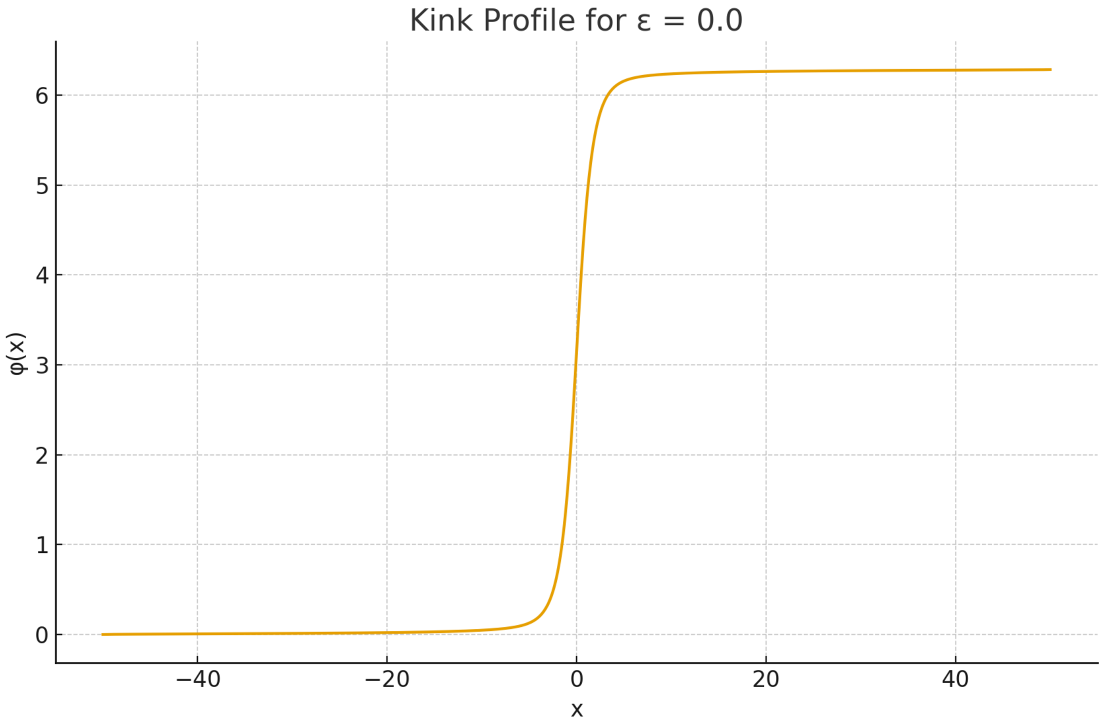

Figure 1: Deformed sine-Gordon kink profile for ϵ =0

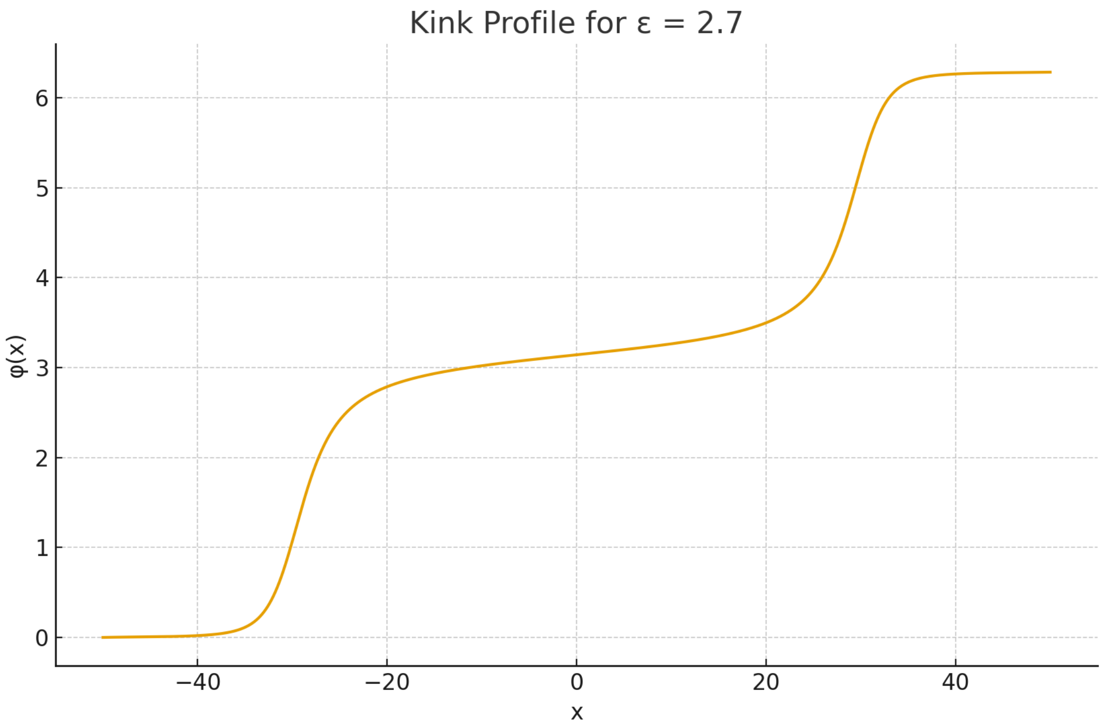

Figure 2: Deformed sine-Gordon kink profile for ϵ =2.7

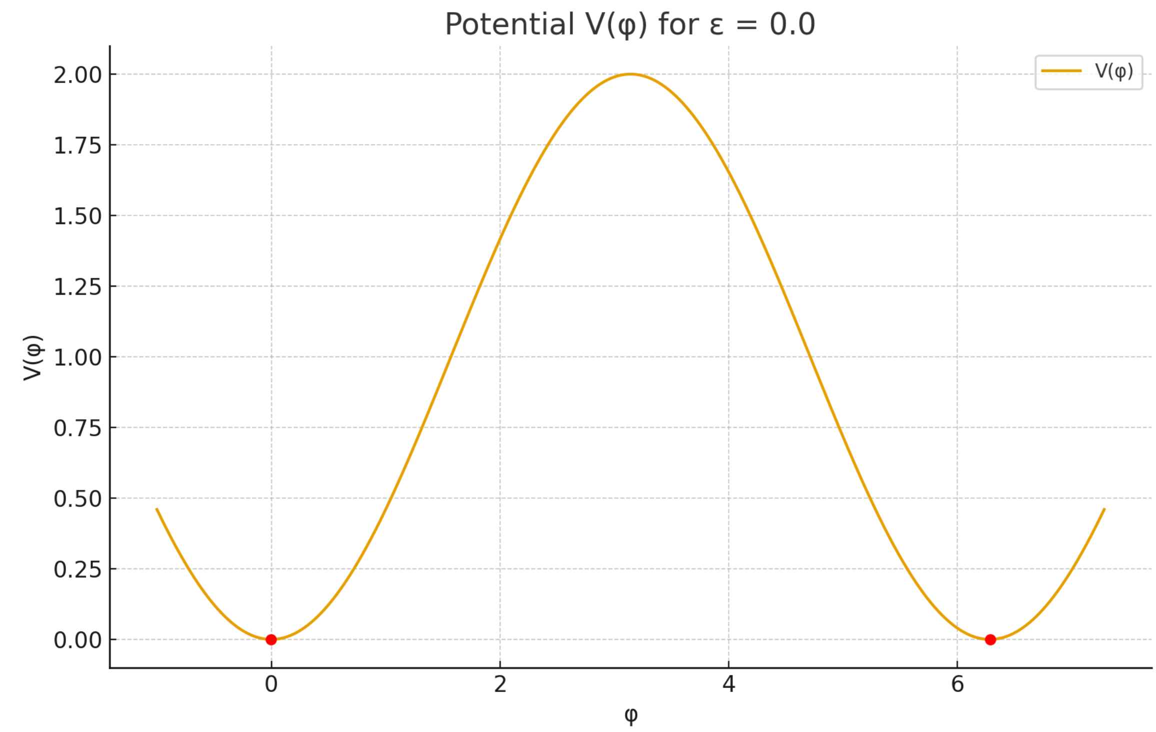

Figure 3: Deformed sine-Gordon potential for ϵ =0

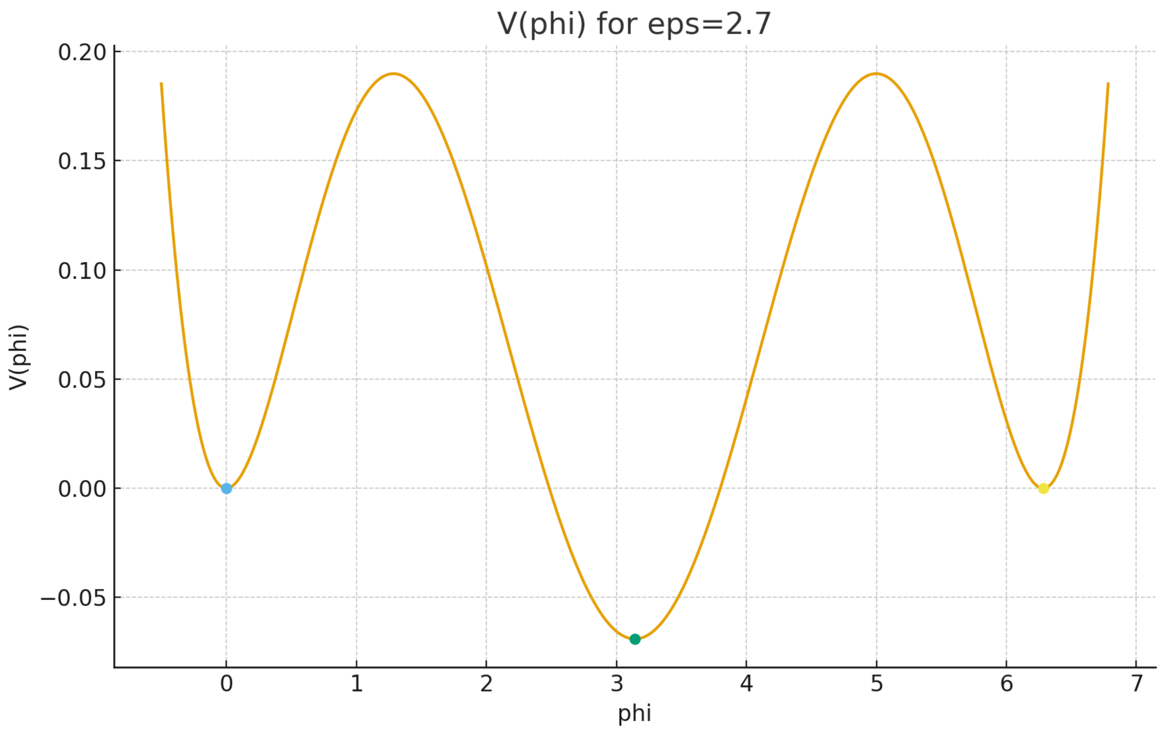

Figure 4: Deformed sine-Gordon potential for ϵ =2.7

References

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