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FINAL REPORT ON THE INTEREST PROGRAMM

Particle production in high-energy collision and statistical mechanics

Supervisor:

Dr. Trambak Bhattacharyya

Student:

Luis Alberto González Flores

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Abstract

The transverse momentum distribution of particles produced in proton-proton and heavy-ion collisions is usually described by the power-law function. Recently, it has been shown that the phenomenological distribution of the Tsallis transverse momentum distribution adequately describes the experimental data for collisions at relativistic energies such as those obtained in laboratories such as the LHC and RHIC. The exact results for this distribution are presented as a series expansion using integral expressions for the Tsallis statistics (normalized and unnormalized). Zero-term approximation expressions for the Tsallis particle statistics are also obtained.

1 Introduction

Collisions between proton-proton or lead-nuclei provide insight into the properties of nuclear matter and its constituents. At higher energies, quarks and gluons inside the nucleons interact and create a short-lived medium called quark-gluon plasma. Also, before the creation of the medium, highly energetic particles (quarks, and gluons) called 'jets' are also created and pass through the medium. After a certain time, the quarks and gluons hadronize and create π, K, p, etc. Momentum distributions of these hadrons are experimentally observed, and there have been many attempts to explain these distributions. One such attempt is inspired by the generalized statistical mechanics proposed by C. Tsallis, and in this report, we discuss that.

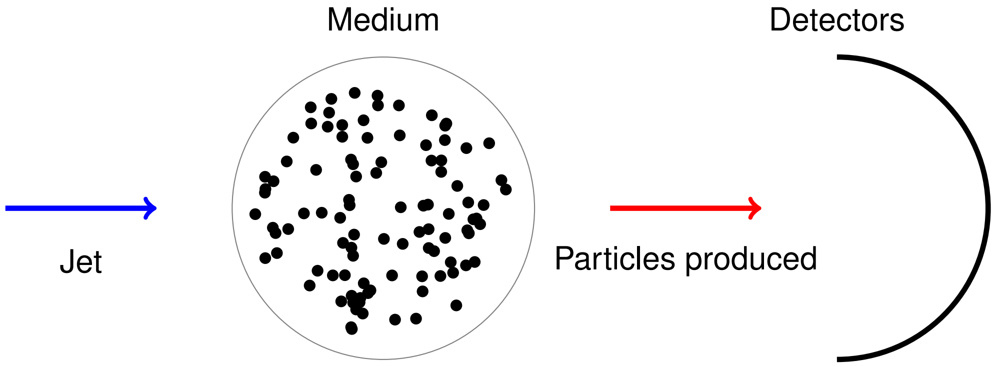

Figure 1: *A jet with a clearly defined direction collides with a medium, producing new particles that are detected.*

2 Participant-spectator model

Consider a "projectile" nucleus that moves in a straight line while interacting with a target nucleus. After the collision, an interaction zone is generated due to the overlap of the two nuclei. This zone is well defined for each impact parameter $\boldsymbol{b}$ which is the transverse distance between the centers of the colliding nuclei. In the interaction zone, the nucleons that interact are referred as 'participants', while the remaining nucleons of the projectile and the target are referred to as 'spectators'. Thus, participants begin to interact with each other while spectators are not significantly affected by the collision and continue with their movement. During the formation and the expansion of the interaction zone, there are enough interactions between the participants to establish a local thermal equilibrium in the zone. Finally, the interaction zone exhibits decay due to emission. When the collision energy is relativistic, the interaction zone generated by the participants create a QGP [1]. The figure 2 is a diagram of the participant-spectator model before and after the collision.

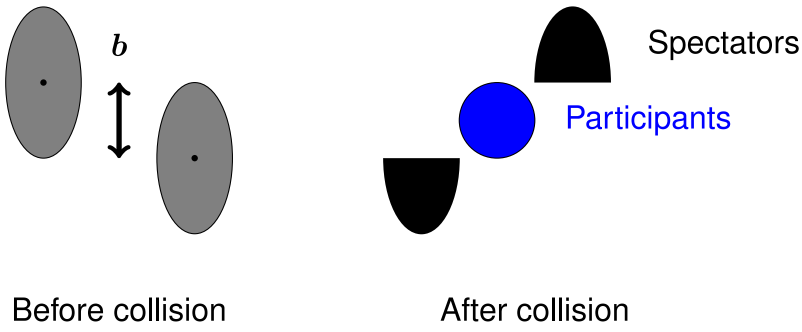

Figure 2: *Participant-spectator model*

3 Basics of QGP

Since the relativistic energy collisions of interest lead to the formation of a QGP, we will briefly describe the basic properties and formation of this state of matter.

At high densities or temperatures, hadronic matter loses its identity and is better described in terms of its constituents: quarks and gluons. Quarks are elementary particles of spin $\frac{1}{2}$ in the Standard Model. There are six types of quarks: *up(u), down(d), strange(s), charm(c), bottom or beauty(b), and top or truth(t)* [1]. These names refer to a certain degree of freedom, known as *flavor*, associated with quarks. However, quarks possess another degree of freedom, called *color*, which is denoted by the letters *r(red), blue(b), and g(green)*. Matter composed is colorless, so it is a specific combination of a certain type of quarks [1]. Gluons are carriers of strong force and they also carry colors.
We can think of QGP formation by imagining a closed system of hadrons. When the density is low, the hadrons are far away from each other.

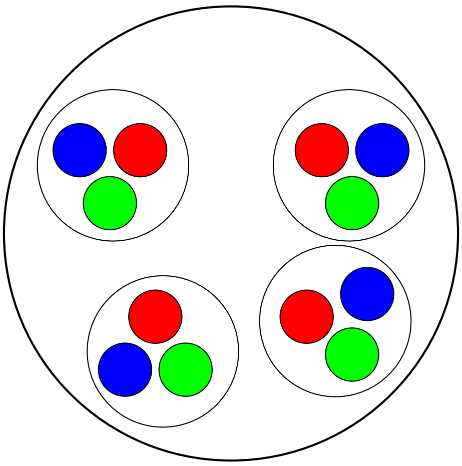

Figure 3: *System of hadrons in low density.*

At high densities and high temperatures, hadrons start to interpenetrate each other, and the constituents become deconfined.

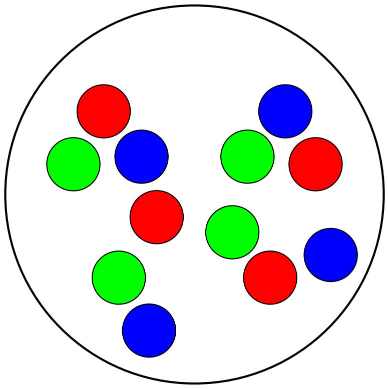

Figure 4: Deconfinement.

As defined by Ref [1], QGP is defined as a thermalized, or near thermalized, state of quarks and gluons, where they are free to move in a nuclear volume rather than a nucleonic volume. The study of this type of matter is crucial in high-energy physics because theoretical models indicate that matter can only exist as a QGP beyond a critical temperature $T_{cr} \sim 200$ MeV.
QGP is the deconfined state of strongly interacting matter. At low densities or low temperatures, quarks are confined within hadrons. However, as the temperatures or density increases, the quarks are no longer confined within the hadrons, resulting in a deconfined state.This is known as confinement-deconfinement phase transition [1].
The importance of studying the QGP lies in the fact that it gives us access to the ancient microsecond universe, since the QGP must have existed in the very early universe. However, the phase transition of matter between these confined and deconfined states can be created at high energies in laboratories. In addition to the above, the study of QGP has given us a new perspective regarding the study of condensed matter physics, which is expanding to new domains [1].

In high-energy collisions, their constituents mix and enter a pre-equilirbium phase before forming a QGP. The QGP has a very short lifetime of around 10^{-23} s. The evolution of this plasma is governed by the hydrodynamic transport equation. When equilibrium is reached, quarks and gluons are in a deconfined state. Subsequently, as the system expands, its density decreases and it cools down, causing the quarks and gluons to undergo hadronisation, i.e. to form hadrons. When the average distance between hadrons exceeds the range of strong interaction, the hadrons decouple and freeze out [1].

This is known as *kinetic freeze-out*. Hadrons from the kinetic freeze-out surface will be detected in the detector.

4 Particle Spectrum, number density and Boltzmann-Gibbs distribution

For hadron collisions, the average number of particles[1] detected in each collision is given by the expression:

$$N = \frac{g}{(2\pi)^3} \int d^3p \, d^3x \, f_{sp}, \tag{1}$$

[1]Natural units are used throughout the text: $c = \hbar = k_B = 1$.

where the parameter g indicates the degeneracy [2] and f_{sp} is the single particle distribution function, e.g. the Boltzmann-Gibbs distribution function

$$f_{sp} = \frac{1}{e^{\beta E} \pm 1}, \tag{2}$$

with $\beta = \frac{1}{T}$ the Gibbs parameter. In the case where the distribution function is a homogeneous function, the expression for N takes the form

$$N = \frac{gV}{(2\pi)^3} \int d^3p \, f_{sp} \quad \Longrightarrow \quad E\frac{dN}{d^3p} = \frac{gV}{(2\pi)^3} E f_{sp}. \tag{3}$$

Since

$$\boldsymbol{p}^2 = p_x^2 + p_y^2 + p_z^2 = p_T^2 + p_z^2, \tag{4}$$

the differential $d^3p = dp_x dp_y dp_z$ can be written as $d^3p = d^2p_T dp_z$, so that we can write

$$E\frac{dN}{d^2p_T dp_z} = \frac{gV}{(2\pi)^3} E f_{sp}, \tag{5}$$

where p_T and p_z are the transverse and longitudinal momentum, respectively, at the collision, and E is given by the relativistic relation

$$E^2 = \boldsymbol{p}^2 + m^2 = p_T^2 + p_z^2 + m^2 = m_T^2 + p_z^2, \tag{6}$$

with m_T the transverse mass, which is a scalar, so what is a Lorentz invariant[3]. The latter allows us to write the following relation

$$E^2 - p_z^2 = m_T^2, \tag{7}$$

where we can choose a suitable parameterization for the variables E and p_z, namely

$$E = m_T \cosh y, \tag{8}$$

$$p_z = m_T \sinh y, , \tag{9}$$

where the parameter y is called *rapidity*. With this parameterization, equation (5) is expressed in the following form

$$\frac{EdN}{d^2p_T E dy} = \frac{gV}{(2\pi)^3} E f_{sp}, \tag{10}$$

which can be integrated with respect to an azimutal angle ϕ $(0 \leq \phi \leq 2\pi)$ by expressing d^2p_T in polar coordinates, which finally leads to the following result

$$\frac{dN}{p_T dp_T dy} = \frac{gV}{(2\pi)^2} E f_{sp}. \tag{11}$$

It is important to note that the right side of the equation (11) represents the theoretical development. This result should be compared to the observed quantity in the experiments, which is located on the left side of the equality. In the case of the Boltzmann-Gibbs distribution, equation (11) takes the form

$$\frac{dN}{p_T dp_T dy} = \frac{gV}{(2\pi)^2} m_T \cosh y e^{-\beta m_T coshy}. \tag{12}$$

However, according the experiments, the Boltzmann-Gibbs statistics does not describe the experimental data. The Boltzmann-Gibbs statistics uses a definition for entropy [2], which is the familiar expression

$$S_{BG} = -\sum_i p_i \ln(p_i). \tag{13}$$

[2]For example, for pions, the value of g is 2 (for π^+ and π^-). In contrast, for protons, the value of g is 4 (two for $p^\pm$) .
[3]In the case where the boost is along the z-axis.

Since entropy is not a universal function, Boltzmann-Gibbs statistics only describe a certain number of systems with certain characteristics. The collisions of interest are not described by this statistic, as evidence by experimental data. Therefore, other type of statistic must be used to described the spectrum of particles produced during collisions. The distribution of transverse momentum of particles produced proton-proton and heavy-ions collisions is described using Power-law function based on the Tsallis statistics. These functions provide an objective and precise description of the phenomenon. Below is a description of Tsallis statistic based on the expression proposed by C. Tsallis for entropy. This expression is used to derive the probabilities in this statistic and the correct expressions for the Tsallis transverse momentum spectrum.

5 Tsallis statistics

In 1988, Constantino Tsallis proposed a generalization to Boltzmann-Gibbs statistical mechanics, now known as Tsallis statistical mechanics. In this generalization, the quantity of interest is p_i^q, where p_i is the probability associated with an event and q is a real number such that $0 \leq q \leq \infty$. This quantity was used by Tsallis to generalize the definition of entropy to a functional from the parameter q [3].

According to the generalization of Tsallis [3], it is postulated that the entropy is expressed as follows

$$S = \sum_{i=1}^{N} \frac{p_i^q - p_i}{1-q}, \tag{14}$$

where the parameter q is a positive number, N is the number of the possible configurations (microstates) of the system[4] and the probabilities p_i are normalized to unity, i.e.

$$\sum_{i=1}^{N} p_i = 1. \tag{15}$$

Within the grand canonical ensemble framework, we discuss two types of Tsallis statistics: normalized or Tsallis-1 statistics and unnormalized or Tsallis-2 statistics. The difference between these statistics is in how the standard expectation values of the observables are defined. We will cover the general formalism for both statistics to calculate the normalized probabilities p_i, and determine the expressions for the particle transverse momentum distribution.

The following detailed study of Tsallis statistics follows Ref [4].

5.1 Tsallis-1 in grand canonical ensemble

In this case, the generalized expectation values for the observables are defined as follows.

$$\langle A \rangle = \sum_i p_i A_i. \tag{16}$$

In the grand canonical ensemble scheme, the thermodynamic potential is

$$\begin{aligned} \Omega &= \langle H \rangle - TS - \mu \langle N \rangle \\ &= \sum_i p_i \left[E_i - \mu N_i - T \frac{p_i^{q-1} - 1}{1-q} \right], \end{aligned} \tag{17}$$

[4]Under the assumption that the spectrum of the system is discrete.

where $\langle H\rangle = \sum_i p_i E_i$ is the mean energy of the system, $\langle N\rangle = \sum_i p_i N_i$ is the mean of particles and E_i and N_i are the energy and number of particles, respectively, in the i-th microstate of the system. Now, to determine the analytical form of the probabilities p_i, we must remember that in the grand canonical ensemble, the set of these probabilities is determined by the local extremes (restricted by the normalization condition (15)) of the thermodynamic potential (17). As the method of the Lagrange multipliers [2] is utilized to determine extrema, we begin by defining the following function:

$$\Phi = \Omega - \lambda\varphi, \tag{18}$$

where

$$\varphi = \sum_i p_i - 1 \tag{19}$$

is a additional function and λ is an arbitrary real constant. The local extrema of the function Φ are determined by the equation

$$\frac{\partial \Phi}{\partial p_j} = 0. \tag{20}$$

By substituting equations (17) and (19) into equation (20), we have

$$\begin{aligned}\frac{\partial \Phi}{\partial p_j} &= \frac{\partial}{\partial p_j}\left(\sum_i p_i\left[E_i - \mu N_i - T\frac{p_i^{q-1}-1}{1-q}\right] - \lambda\left[\sum_i p_i - 1\right]\right)\\ &= \sum_i\left(-\frac{T(q-1)p_i^{q-1}}{1-q} + \left[E_i - \mu N_i - T\frac{p_i^{q-1}-1}{1-q}\right] - \lambda\right)\delta_{ij} = 0,\end{aligned} \tag{21}$$

given that $\frac{\partial E_i}{\partial p_j} = \frac{\partial N_i}{\partial p_j} = 0$. Thus

$$Tp_j^{q-1}\left(1 - \frac{1}{1-q}\right) = \lambda - E_j + \mu N_j + \frac{T}{q-1}, \tag{22}$$

which gives

$$\begin{aligned}p_j^{q-1} &= \frac{q-1}{q}\frac{\Lambda - E_j + \mu N_j}{T} + \frac{1}{q} + \frac{q-1}{q}\\ &= 1 + \frac{q-1}{q}\frac{\Lambda - E_j + \mu N_j}{T},\end{aligned} \tag{23}$$

where $\Lambda = \lambda - T$. Therefore

$$p_i = \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}} \quad\Longrightarrow\quad \sum_i\left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}} = 1. \tag{24}$$

When $q \to 1$, the Gibbs probability distribution is recovered:

$$p_i = \exp\left(\frac{\Lambda - E_i + \mu N_i}{T}\right). \tag{25}$$

A comparison of this expression with the probabilities for the grand canonical ensemble reveals that the partition function is

$$Z = \sum_i \exp[-(E_i - \mu N_i)/T]. \tag{26}$$

Given the expression for the probabilities, we can explicitly write the expectation values of the observables as follows:

$$\langle A\rangle = \sum_i A_i \left[1 + \frac{q-1}{q}\frac{\Lambda - E_i + \mu N_i}{T}\right]^{\frac{1}{q-1}}. \tag{27}$$

To express the probabilities and normalization condition (written in (24), as well as the expectation values (27), it is convenient to use an integral representation with the help of the following expressions for the integral representation of the Gamma function [5, 6]:

$$x^{-y} = \frac{1}{\Gamma(y)}\int_0^\infty t^{y-1}e^{-tx}dt, \quad \text{Re}(y) > 0, \tag{28}$$

$$x^{y-1} = \Gamma(y)\frac{i}{2\pi}\oint_C (-t)^{-y}e^{-tx}dt, \quad \text{Re}(x) > 0, |y| < \infty. \tag{29}$$

Thus, for $q < 1$, the probabilities are as follows:

$$p_i = \frac{1}{\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{1}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-E_i+\mu N_i}{T}\right]}dt, \tag{30}$$

while for $q > 1$, the probabilities are expressed as:

$$p_i = \Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-E_i+\mu N_i}{T}\right]}dt. \tag{31}$$

The equation for the norm is expressed as follows:

$$\frac{1}{\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{1}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}dt = 1 \quad \text{for} \quad q < 1, \tag{32}$$

and

$$\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}dt = 1 \quad \text{for} \quad q > 1, \tag{33}$$

where $\beta' = t(1-q)/qT$, and $\Omega_G(\beta') = -\frac{1}{\beta'}Z_G(\beta')$ with $Z = \sum_i(\beta') = \sum_i e^{\beta'(E_i - \mu N_i)}$, and the expectation values take the following form in this integral representation

$$\langle A\rangle = \frac{1}{\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')dt \quad \text{for} \quad q < 1 \tag{34}$$

and

$$\langle A\rangle = \Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')dt \quad \text{for} \quad q > 1, \tag{35}$$

where

$$\langle A\rangle_G(\beta') = \frac{1}{Z_G(\beta')}\sum_i A_i e^{-\beta'(E_i - \mu N_i)}. \tag{36}$$

The equations (34) and (35) link the statistical averages in the formalism of the Tsallis-1 statistics with the corresponding statistical averages of the Boltzmann-Gibbs statistics in equation (36) [4].

Transverse momentum distribution in Tsallis-1 statistics

From the expressions for the statistical averages in Tsallis statistics 1, we can write the corresponding expressions for the transverse momentum distribution in the grand canonical ensemble scheme. In the case of an ideal gas [4], these expression can be expanded into a series as

$$\frac{dN}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)} \times \int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} \frac{(-\beta'\Omega_G(\beta'))^n}{e^{\beta'(m_T\cosh y-\mu)}+\eta} dt \tag{37}$$

for $q < 1$

and

$$\frac{dN}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{q}{q-1}\right)}{n!} \frac{i}{2\pi} \times \oint_C (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} \frac{(-\beta'\Omega_G(\beta'))^n}{e^{\beta'(m_T\cosh y-\mu)}+\eta} dt \tag{38}$$

for $q > 1,$

where p_T is the tranverse momentum, m_T is the tranverse mass, and y is the rapidity. The norm function Λ can be calculated through equation (24) which in the grand canonical ensemble for both quantum and classical statistics of particles in integral representations (equations (32) and (33)) can be expanded into a series as

$$\sum_0^\infty \frac{1}{n!\Gamma\left(\frac{1}{1-q}\right)} \int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} (\beta'\Omega_G(\beta'))^n dt = 1 \quad \text{for} \quad q < 1 \tag{39}$$

and

$$\sum_0^\infty \frac{\Gamma\left(\frac{q}{q-1}\right)}{n!} \frac{i}{2\pi} \oint (-t)^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} (\beta'\Omega_G(\beta'))^n dt = 1 \quad \text{for} \quad q > 1, \tag{40}$$

with

$$-\beta'\Omega_G(\beta') = \sum_{\boldsymbol{p},\sigma} \ln\left[1+\eta e^{-\beta'(\varepsilon_{\boldsymbol{p}}-\mu)}\right]^{\frac{1}{\eta}}, \tag{41}$$

what is the thermodynamic potential for an ideal gas in Boltzmann-Gibbs statistics whose energy per particle is $\varepsilon_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2}$ and m is the mass of a particle.

Maxwell-Botzmann statistics of particles

In the case of Maxwell-Boltzmann statistics, we consider the limit $\eta \to 0$ where the thermodynamic potential of the ideal gas of the Boltzmann-Gibbs statistics (41) takes the form

$$\Omega_G(\beta') = -\frac{gV}{(2\pi)^2}\frac{m^2}{\beta'^2} e^{\beta'\mu} K_2(\beta' m), \tag{42}$$

where K_ν is the modified Bessel function of the second kind. Substituting this thermodynamic potential into equations (39) and (40), and taking the limit $\eta \to 0$, we obtain the norm equation for the Maxwell-Boltzmann statistics of particles as

$$\begin{aligned}\sum_0^\infty \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \int_0^\infty t^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda+\mu n}{T}\right]} \\ \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \quad \text{for} \quad q<1\end{aligned} \tag{43}$$

which can be briefly written as

$$\sum_0^\infty \phi(n) = 1 \tag{44}$$

and

$$\begin{aligned}\sum_0^\infty \frac{-(\omega)^n}{n!} \Gamma\left(\frac{q}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda+\mu n}{T}\right]} \\ \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \quad \text{for} \quad q>1\end{aligned} \tag{45}$$

where

$$\omega = \frac{gVTm^2}{2\pi^2} \frac{q}{1-q}. \tag{46}$$

Then, by substituting equation (24) into the equations for the transverse momentum distribution ((37) and (38)) and taking the limit $\eta \to 0$, we obtain the transverse momentum distribution for the Maxwell-Boltzmann statistics [4]. These expressions are as follows:

$$\begin{aligned}\frac{dN}{dp_T dy} =& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^\infty \frac{\omega^n}{n!} \frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \\ &\times \int_0^\infty t^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-m_T\cosh y+\mu(n+1)}{T}\right]} \\ &\times \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \quad \text{for} \quad q<1\end{aligned} \tag{47}$$

and

$$\begin{aligned}\frac{dN}{dp_T dy} =& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^\infty \frac{(-\omega)^n}{n!} \Gamma\left(\frac{q}{q-1}\right) \\ &\times \frac{i}{2\pi} \oint_C (-t)^{\frac{q}{1-q}-n} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda-m_T\cosh y+\mu(n+1)}{T}\right]} \\ &\times \left(K_2\left(\frac{t(1-q)m}{qT}\right)\right)^n dt \quad \text{for} \quad q>1.\end{aligned} \tag{48}$$

Zeroth term aproximation

I we kepp only the zeroth order term ($n = 0$) in the series expansion of equation (39) and (40), we obtain

$$\frac{1}{\Gamma\left(\frac{1}{1-q}\right)} \int_0^\infty t^{\frac{q}{1-q}} e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]} dt = \left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]^{\frac{1}{q-1}} = 1 \quad \text{for} \quad q<1, \tag{49}$$

and

$$\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C(-t)^{\frac{q}{1-q}}e^{-t\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]}dt=\left[1+\frac{q-1}{q}\frac{\Lambda}{T}\right]^{\frac{1}{q-1}}=1 \quad \text{for} \quad q>1, \tag{50}$$

where we have used equations (28) and (29). This tells us that the norm function Λ is zero in this order. Substituting this value into the equations for the distribution of the transverse momentum (37) and (38) and considering only the zeroth term, we obtain

$$\begin{aligned}\frac{dN}{dp_T dy}=&\frac{gV}{(2\pi)^2}p_T m_T\cosh y\\&\times\frac{1}{\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{q}{1-q}}e^{-t}\frac{1}{e^{\beta'(m_T\cosh y-\mu)}+\eta}dt\end{aligned} \tag{51}$$

for $q<1$

and

$$\begin{aligned}\frac{dN}{dp_T dy}=&\frac{gV}{(2\pi)^2}p_T m_T\cosh y\\&\times\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C(-t)^{\frac{q}{1-q}}e^{-t}\frac{1}{e^{\beta'(m_T\cosh y-\mu)}+\eta}dt\end{aligned} \tag{52}$$

for $q>1.$

According to the following equation

$$\frac{1}{e^x+\eta}=\sum_{k=0}^{\infty}(-\eta)^k e^{-x(k+1)}, \quad \text{where} \quad |e^{-x}|<1, \tag{53}$$

the above expressions can be rewritten as

$$\begin{aligned}\frac{dN}{dp_T dy}=&\frac{gV}{(2\pi)^2}p_T m_T\cosh y\\&\times\sum_{k=0}^{\infty}(-\eta)^k\frac{1}{\Gamma\left(\frac{1}{1-q}\right)}\int_0^\infty t^{\frac{q}{1-q}}e^{-t\left[1+(k+1)\frac{1-q}{q}\frac{m_T\cosh y-\mu}{T}\right]}dt\\=&\frac{gV}{(2\pi)^2}p_T m_T\cosh y\\&\times\sum_{k=0}^{\infty}(-\eta)^k\left[1+(k+1)\frac{1-q}{q}\frac{m_T\cosh y-\mu}{T}\right]^{\frac{1}{q-1}} \quad \text{for} \quad q<1,\end{aligned} \tag{54}$$

and

$$\begin{aligned}\frac{dN}{dp_T dy}=&\frac{gV}{(2\pi)^2}p_T m_T\cosh y\\&\times\sum_{k=0}^{\infty}(-\eta)^k\Gamma\left(\frac{q}{q-1}\right)\frac{i}{2\pi}\oint_C(-t)^{\frac{q}{1-q}}e^{-t\left[1+(k+1)\frac{1-q}{q}\frac{m_T\cosh y-\mu}{T}\right]}dt\\=&\frac{gV}{(2\pi)^2}p_T m_T\cosh y\\&\times\sum_{k=0}^{\infty}(-\eta)^k\left[1+(k+1)\frac{1-q}{q}\frac{m_T\cosh y-\mu}{T}\right]^{\frac{1}{q-1}} \quad \text{for} \quad q>1,\end{aligned} \tag{55}$$

considering equations (28) and (29) and using $\beta' = (1-q)qT$. For both cases ($q < 1$ and $q > 1$) the same expression for the distribution of the transverse momentum is obtained, which is valid for the values $\eta = -1, 0, 1$. In particular, in the case of the Maxwell-Boltzmann statistic for particles ($\eta = 0$), the transverse momentum distribution at order zero is

$$\frac{dN}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1 + \frac{1-q}{q}\frac{m_T \cosh y - \mu}{T}\right]^{\frac{1}{q-1}} . \tag{56}$$

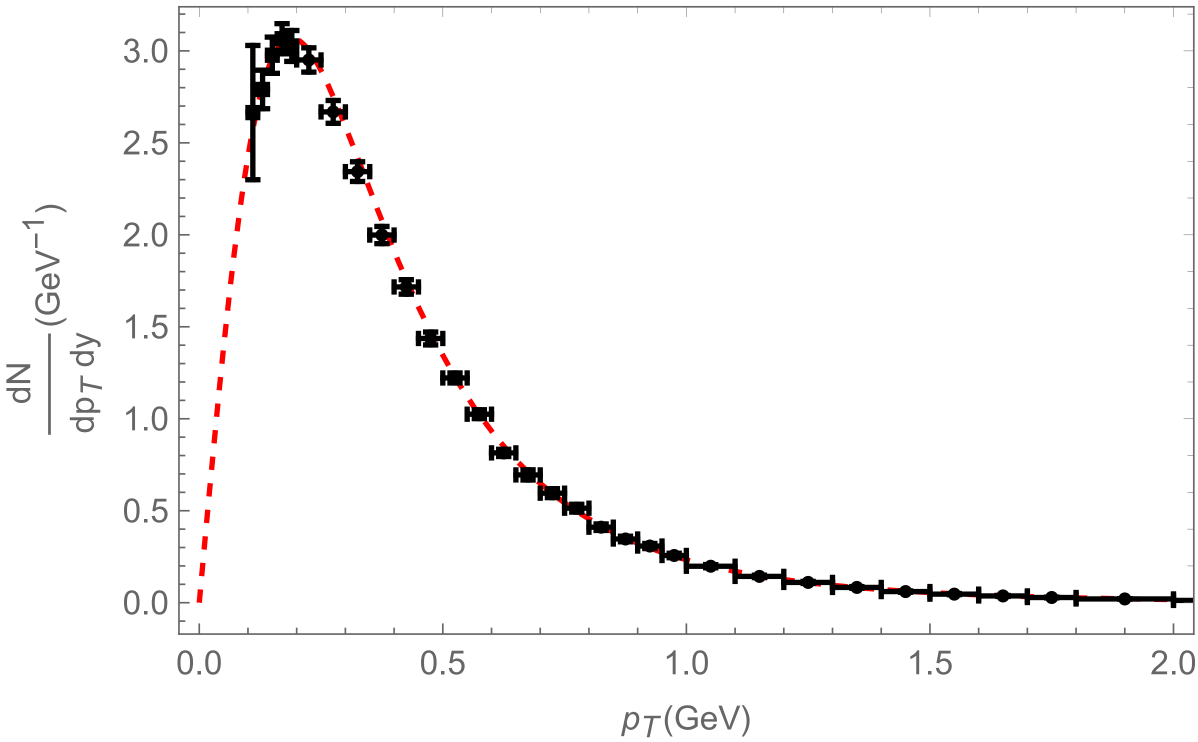

Figure 5: Plot of the zeroth order function of the transverse momentum distribution for the π^--particles produced in p-p collisions obtained by the ALICE Collaboration at $\sqrt{s} = 0.9$ TeV [7]. The parameters [4] of the Tsallis-1 statistical fit are: rapidity $y = 0$, temperature $T = 71.837$ MeV, chemical potential $\mu = 0$, radius $R = 4.743$ fm, entropic parameter $q = 0.873$ and mass $m = 139.57$ MeV (pion mass).

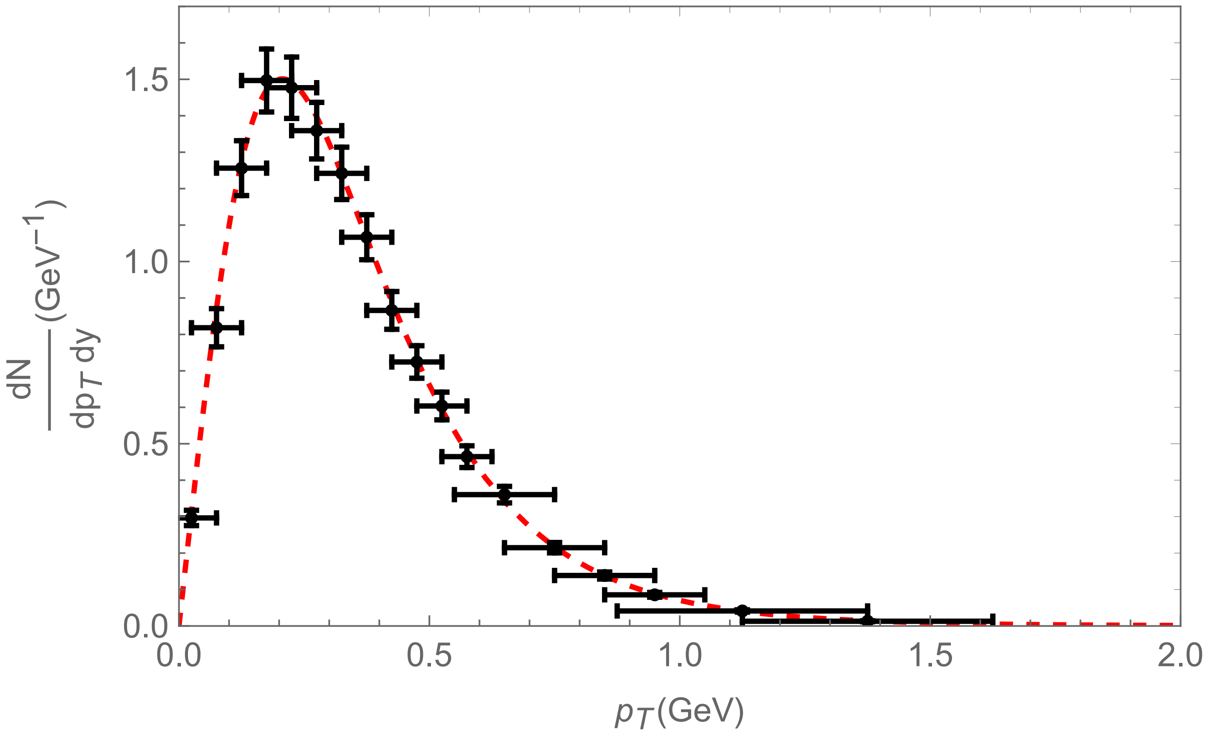

Figure 6: Plot of the zeroth order function of the transverse momentum distribution for the π^-particles produced in the p-p collisions obtained by the NA61/SHINE Collaboration at $\sqrt{s} = 17.3$ GeV [8]. The parameters of the Tsallis-1 statistical fit are [4]: rapidity $y = 0$, temperature $T = 90.574$ MeV, chemical potential $\mu = 0$, radius $R = 3.140$ fm, entropic parameter $q = 0.931$ and mass $m = 139.57$ MeV (pion mass).

5.2 Tsallis-2 statistics in the grand canonical ensemble

The Tsallis-2 statistics, or Tsallis unnormalized statistics, uses the same definition of entropy as the Tsallis-1 statistics with the probabilities p_i of the microstates normalized to unity. However, as mentioned above, the difference between these statistics lies in the definition of the expectation values, which in this case are

$$\langle A \rangle = \sum_i p_i^q A_i. \tag{57}$$

In this case, the thermodynamic potential Ω of the grand canon ensemble takes the form

$$\begin{aligned} \Omega &= \langle H \rangle - Ts - \mu N \\ &= \sum_i p_i^q \left[E_i - \mu N_i + T \frac{p_i^{1-q} - 1}{1-q} \right], \end{aligned} \tag{58}$$

where the expression for entropy is conveniently rewritten as

$$S = -\sum_i p_i \left(\frac{p_i^{1-q} - 1}{1-q} \right). \tag{59}$$

Analogously to the case of Tsallis-1 statistics, using the method of the Lagrange multipliers [2], we can find the expression for the probabilities p_i, which are given by the equation

$$\frac{\partial \varphi}{\partial p_j} = 0, \tag{60}$$

where

$$\varphi = \Omega - \lambda \left(\sum_i -1 \right). \tag{61}$$

The method of the Lagrange multipliers leads us to the fact that in this case the probabilities p_i have the following form

$$p_i = \frac{1}{Z} \left[1 - (1-q) \frac{E_i - \mu N_i}{T} \right]^{\frac{1}{1-q}}, \tag{62}$$

with

$$Z = \sum_i \left[1 - (1-q) \frac{E_i - \mu N_i}{T} \right]^{\frac{1}{1-q}}, \tag{63}$$

where $Z \equiv= [(1-(1-q)\lambda/T)/q]^{\frac{1}{1-q}}$ which is the norm function. Like the partition function, it is related to the Lagrange multiplier λ. As with the probabilities, norm function, and expectation values for the Tsallis-1 statistics, we write these results in an integral representation for the Tsallis-2 statistics. Thus, using equations (28) and (29), we can write for the probabilities

$$p_i = \frac{1}{Z} \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}-1} e^{-t\left[1-(1-q)\frac{E_i-\mu N_i}{T}\right]} dt \tag{64}$$

$$\text{for} \quad q > 1,$$

and

$$p_i = \frac{1}{Z} \Gamma\left(\frac{2-q}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-1} e^{-t\left[1-(1-q)\frac{E_i-\mu N_i}{T}\right]} dt \tag{65}$$

$$\text{for} \quad q < 1,$$

so the partition function in the integral representation takes the following form

$$Z = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}-1} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]} dt \tag{66}$$

$$\text{for} \quad q > 1$$

and

$$Z = \Gamma\left(\frac{2-q}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-1} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]} dt \tag{67}$$

$$\text{for} \quad q > 1,$$

wehre $\beta' = t(q-1)/T$ and $\Omega_G(\beta') = \frac{1}{\beta'} Z_G(\beta')$. Equations (64) and (65), as equations (30) and (31), are the link between the probability distributions of the Tsallis-2 statistics and the probability distributions of the Boltzmann-Gibbs statistics, while the equations (66) and (67) link the partition function in Tsallis-2 statistics with the thermodynamic potential of the Boltzmann-Gibbs statistics [4]. In the integral representation, the statistical averages of observables are

$$\langle A \rangle = \frac{1}{Z^q} \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^\infty t^{\frac{1}{q-1}} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]} \langle A \rangle_G(\beta') dt \tag{68}$$

$$\text{for} \quad q > 1,$$

and

$$\langle A\rangle = \frac{1}{Z^q}\Gamma\left(\frac{1}{1-q}\right)\frac{i}{2\pi}\oint_C (-t)^{\frac{1}{q-1}} e^{-t\left[1-(1-q)\frac{\Omega_G(\beta')}{T}\right]}\langle A\rangle_G(\beta')dt \tag{69}$$

for $q < 1,$

where $\langle A\rangle_G(\beta')$ are the statistical averages of Boltzmann-Gibbs statistics defined in equation (36).

Transverse momentum distribution in the Tsallis-2 statistics

Given the expressions (68) and (69) for the statistical averages, the distribution of the transverse momentum in the Tsallis-2 statistic is obtained in this case. For an ideal gas [4], this distribution is expressed as

$$\begin{aligned}\frac{dN}{dp_T dy} =& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty}\frac{1}{n! Z^q \Gamma\left(\frac{q}{q-1}\right)}\\ &\int_0^{\infty} t^{\frac{1}{q-1}} e^{-t}\frac{(\beta'\Omega_G(\beta'))^n}{e^{\beta'(m_T\cosh y-\mu)}+\eta}dt\end{aligned} \tag{70}$$

for $q > 1,$

and

$$\begin{aligned}\frac{dN}{dp_T dy} =& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{1-q}\right)}{n! Z^q}\frac{i}{2\pi}\\ &\oint_C (-t)^{\frac{1}{q-1}} e^{-t}\frac{(\beta'\Omega_G(\beta'))^n}{e^{\beta'(m_T\cosh y-\mu)}+\eta}dt\end{aligned} \tag{71}$$

for $q < 1,$

where the values $\eta = 1, 0, -1$ correspond to the Fermi-Dirac, the Maxwell-Boltzmann, and the Bose-Einstein statistics, respectively [4]. Regarding the partition function Z, their integral representation can be expanded into a series as

$$Z = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma\left(\frac{1}{q-1}\right)}\int_0^{\infty} t^{\frac{1}{q-1}-1} e^{-t}(\beta'\Omega_G(\beta'))^n dt \tag{72}$$

for $q > 1,$

and

$$Z = \sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{2-q}{1-q}\right)}{n!}\frac{i}{2\pi}\oint_C (-t)^{\frac{1}{q-1}-1} e^{-t}(\beta'\Omega_G(\beta'))^n dt \tag{73}$$

for $q < 1.$

It is worth noting that by taking $n = 0$ and by virtue of equations (28) and (29), the partition function is $Z = 1$. This result will be useful in computing the zeroth term approximation of the transverse momentum distribution in the Tsallis-2 statistics.

Maxwell-Boltzmann statistics of particles

As in the case of the Tsallis-1 statistics, the expressions for the Maxwell-Boltzmann statistics of particles can be written explicitly by taking the limit $\eta \to 0$ [4]. Substituting equation (42) into the above expressions for the partition function, we obtain

$$Z = \sum_{n=0}^{\infty} \frac{\omega^n}{n!\Gamma\left(\frac{1}{q-1}\right)} \times \int_0^{\infty} t^{\frac{1}{q-1}-1} e^{-t\left[1+(1-q)\frac{\mu n}{T}\right]} \times \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \tag{74}$$

$$\text{for} \quad q > 1,$$

which can be briefly written as

$$Z = \sum_{0}^{\infty} \psi(n), \tag{75}$$

and

$$Z = \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \Gamma\left(\frac{2-q}{1-q}\right) \frac{i}{2\pi} \times \oint_C (-t)^{\frac{1}{q-1}-1-n} e^{-t\left[1+(1-q)\frac{\mu n}{T}\right]} \times \left(K_2\left(\frac{t(q-1)m}{T}\right)\right)^n dt \tag{76}$$

$$\text{for} \quad q < 1,$$

where

$$\omega = \frac{gVTm^2}{(2\pi)^2} \frac{1}{q-1}. \tag{77}$$

This expressions are the partition function for the Maxwell-Boltzmann statistics of particles in the Tsallis-2 statistics [4]. In the same way as in equation (42), and taking the limit $\eta \to 0$ in equations (70) and (71), we obtain the transverse momentum distribution for the Maxwell-Boltzmann statistics of particles in the Tsallis unnormalized or Tsallis-2 statistics, namely

$$\frac{dN}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{\omega^n}{n!} \frac{1}{Z^q} \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \times \int_0^{\infty} t^{\frac{1}{q-1}-n} e^{-t\left[1-(1-q)\frac{m_T \cosh y - \mu(n+1)}{T}\right]} \times \left(K_2\left(\frac{t(q-1)m}{T}\right)\right) dt \tag{78}$$

$$\text{for} \quad q > 1,$$

and

$$
\begin{aligned}
\frac{dN}{dp_T dy} =& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{n=0}^{\infty} \frac{(-\omega)^n}{n!} \frac{1}{Z^q} \Gamma\left(\frac{1}{1-q}\right) \\
& \times \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}-n} e^{-t\left[1-(1-q)\frac{m_T \cosh y - \mu(n+1)}{T}\right]} \\
& \times \left(K_2\left(\frac{t(q-1)m}{T}\right)\right) dt
\end{aligned}
\tag{79}
$$

for $\quad q < 1.$

Zeroth term aproximation

With $n = 0$ in equations (70) and (71) and using equation (53), the transverse momentum distribution is

$$
\begin{aligned}
\frac{dN}{dp_T dy} =& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \frac{1}{\Gamma\left(\frac{q}{q-1}\right)} \int_0^{\infty} t^{\frac{1}{q-1}} e^{-t\left[1+(k+1)(q-1)\frac{m_T \cosh y - \mu}{T}\right]} dt \\
=& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \left[1 + (k+1)(q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}
\end{aligned}
\tag{80}
$$

for $\quad q > 1,$

and

$$
\begin{aligned}
\frac{dN}{dp_T dy} =& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \Gamma\left(\frac{1}{1-q}\right) \frac{i}{2\pi} \oint_C (-t)^{\frac{1}{q-1}} e^{-t\left[1+(k+1)(q-1)\frac{m_T \cosh y - \mu}{T}\right]} dt \\
=& \frac{gV}{(2\pi)^2} p_T m_T \cosh y \sum_{k=0}^{\infty} (-\eta)^k \left[1 + (k+1)(q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}}
\end{aligned}
\tag{81}
$$

for $\quad q < 1,$

where $\beta' = t(q-1)/T$ and equations (28) and (29) are used. The expressions (80) and (81) are valid for $\eta = 1, 0, -1$, i.e. for the Fermi-Dirac, Maxwell-Boltzmann, and Bose Einstein statistics of particles, respectively. In the case of $\eta = 0$, the distribution of the transverse momentum at the order of zero has the form

$$
\frac{dN}{dp_T dy} = \frac{gV}{(2\pi)^2} p_T m_T \cosh y \left[1 + (q-1)\frac{m_T \cosh y - \mu}{T}\right]^{\frac{q}{1-q}} \tag{82}
$$

Formally, this result can be derived from the distribution of the transverse momentum in the normalized Tsallis or Tsallis-1 statistics by transforming the parameter $q \to 1/q$ in equation (56) [4].

6 Summary and conclusions

Analytical expressions have been derived for the distribution of transverse momentum in the Tsallis statistics, normalized and unnormalized, within the framework of the grand canonical ensemble. The exact results are presented as a series expansion using integral representations of the distribution. Additionally, we have obtained the expression in the zeroth term approximation of the transverse momentum distribution for the Maxwell-Boltzmann ($\eta = 0$), Fermi-Dirac ($\eta = 1$) and Bose-Einstein

($\eta = -1$) statistics of particles. In addition, the zero-order function was plotted for certain values of the parameters of the Tsallis-1 statistical fit for the transverse momentum distribution obtained for the π^-- particles produced in the $p-p$ collisions by the ALICE Collaboration at $\sqrt{s} = 0.9$ TeV and by the NA61/SHINE Collaboration at $\sqrt{s} = 17.3$ GeV. Furthermore, the discussion of the Tsallis statistics shows that the zeroth term approximation of the unnormalized Tsallis statistics (82) can be obtained from the zeroth term approximation of the Tsallis normalized statistics (56) by transformation of the entropic parameter $q \to 1/q$.

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References

[1] A K Chaudhuri. *A Short Course on Relativistic Heavy Ion Collisions*. 2053-2563. IOP Publishing, 2014. ISBN 978-0-750-31060-4. doi: 10.1088/978-0-750-31060-4. URL https://dx.doi.org/10.1088/978-0-750-31060-4.

[2] R.K. Pathria and Paul D. Beale. *Statistical Mechanics*. Academic Press, 3 2021. ISBN 9780081026939.

[3] Constantino Tsallis. Possible generalization of Boltzmann-Gibbs statistics. *J. Stat. Phys.*, 52(1-2):479–487, July 1988.

[4] A S Parvan and T Bhattacharyya. Hadron transverse momentum distributions of the tsallis normalized and unnormalized statistics. *Eur. Phys. J. A*, 56(2), February 2020.

[5] M. Abramowitz and I. Stegun. *Handbook of Mathematical Functions*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Govt. Printing Office, Washington, DC, 1965.

[6] D. Prato. Phys. Lett. A. 203:165, 1995.

[7] K. Aamodt and et al. (ALICE Collaboration). Eur. phys. J C. 71:1655, 2011.

[8] N. Abgrall and et al. (NA61/SHINE Collaboration). Eur. phys. J. C. 74:2794, 2014.