JOINT INSTITUTE FOR NUCLEAR RESEARCH

Veksler and Baldin laboratory of High Energy Physics

FINAL REPORT ON THE INTEREST PROGRAMME

Puzzles of multiplicity

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1 Abstract

Multiplicity distributions in electron–positron annihilation are analyzed using the Two Stage Model. The quark–gluon jets are described as Markov branching processes the hadronization phase is characterized with binomial distribution. A fit to PETRA experimental data is carried out. In addition, the multiplicity distribution, mean multiplicity, and second correlation moment for three-gluon quarkonium decay are calculated.

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2 Introduction

One of the observables in the high energy physics experiments is the number of secondary particles - multiplicity, especially multiplicity of charged particles. This quantity is usually considered with help of statistical methods.

To understand and describe experiments carried out at accelerators, in particular, multiplicity distributions (MD), transverse and longitudinal momenta and etc., Monte Carlo generators are built. Up to now, Monte Carlo simulation has difficulties in the description of MD in the high multiplicity region. Therefore, new approaches to describe MD are needed.

In accordance with present understanding, the multi-particle production in e^+e^--annihilation occurs through two stages: development of a quark-gluon cascade and hadronization. Schematic representation of this process is shown on figure 1 [1]:

Currently, the mechanism of the transition of quarks and gluons into hadrons is not reliably known. Due to the very low energy scale involved in this transition it is impossible to calculate hadronization in the framework of perturbative QCD. Therefore many phenomenological models have been created to describe the hadronization of partons at the end of the parton shower development.

The study of multiparticle production (multiplicity) can deepen our understanding of how quarks and gluons transform into hadrons and expand our knowledge of quark–gluon matter properties. It also allows us to gain further insight into the nature of strong interactions.

3 Project goals

- Calculation of multiplicity distribution (MD) for charged particles in e^+e^--annihilation in framework of Two Stage Model (TSM).

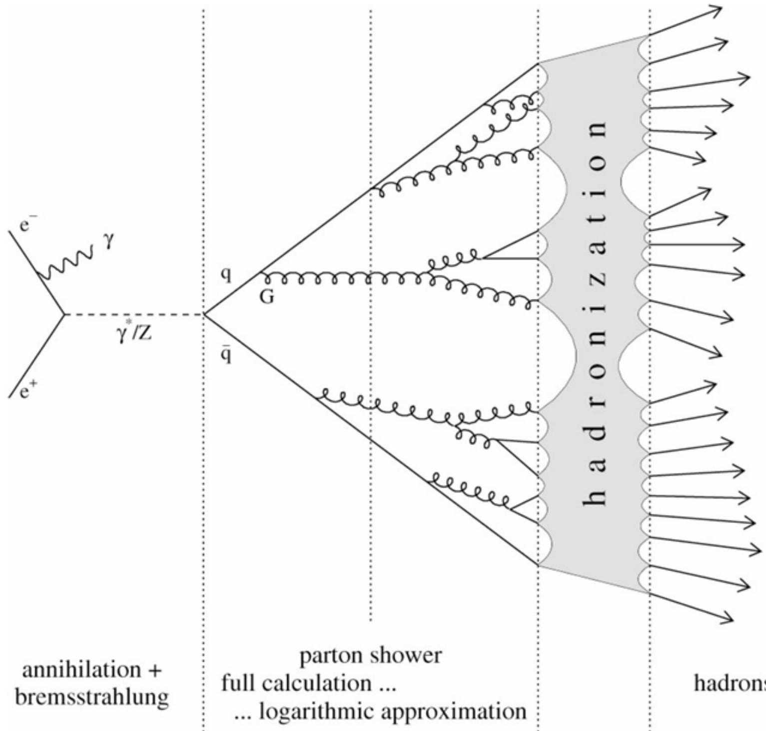

Figure 1: Schematic representation of e^+e^--annihilation process into hadrons.

- Fitting of experimental data on multiplicity distributions by the TSM. Definition of parameters of the model.
- Calculation of MD and it's average multiplicity for upsilon (bottomonium) decays to three gluons with hadronization.

4 Calculation of MD of charged particles in e^+e^--annihilation using TSM

Multplicity is the number of secondaries n in process of multiparticle production:

$$a + b \to c_1 + c_2 + ... + c_n \tag{1}$$

We consider electron-positron annihilation process, because it is simple for analysis, as the produced state is pure $q\overline{q}$. The e^+e^-–annihilation according to QCD can be realized through the production of γ or Z^0-boson:

$$e^-e^+ \to (\gamma, Z^0) \to q\overline{q}$$

Perturbative QCD can only describe the process of parton fission at high energy, the cascade stage, but cannot describe the stage of hadronization. Therefore phenomenological models are used for description of hadronization.

Quark-gluon jets can be described as Markov branching processes [4]. We consider the natural QCD evolution parameter:

$$Y = \frac{1}{2\pi b} \log\left(1 + \alpha b \log\left(\frac{Q^2}{\mu^2}\right)\right), \tag{2}$$

where $2\pi b = (11N_c - 2N_f)/6$ for a theory with N_c colours and N_f flavours, Q-transferred momentum, μ-threshold constant (with the loss of energy quark or gluon goes into μ state). Parameter Y can be interpreted as the thickness value of a quark or gluon which gives origin to a quark (gluon) jet.

Three elementary process which contribute to the quark or gluon distributions into QCD jets are:

1. gluon fission: $g \to g + g$;
2. quark bremsstrahlung: $q \to q + g$;
3. quark pair creation: $g \to q + \overline{q}$.

Let $A\Delta Y$ denote the probability that a gluon in the infinitesimal interval ΔY splits into two gluons, $\tilde{A}\Delta Y$ – the probability that a quark emits a gluon, and $B\Delta Y$ – the probability that a gluon produces a quark–antiquark pair. The coefficients $A, \tilde{A}, B$ are taken to be constants, independent of Y, and each individual parton evolves independently of the others, always with the same infinitesimal probability.

Let us definite the probability that m_g gluons and m_q quarks will be transformed into n_g gluons and n_q quarks over a jet of thickness Y and call it $P_{m_g,m_q;n_g,n_q}(Y)$.

In general case, the probability generating function for a parton jet will be:

$$G^P(u;Y) = \sum_{m=0}^{\infty} P_m^P(Y) u^m, \tag{3}$$

where $P_m^P(Y)$ is the probability that parton will be transformed into m gluons over a jet of Y thickness. So, the probability generating functions for a gluon jet and a quark jet will be, respectively:

$$G(u_g, u_q; Y) = \sum_{n_g,n_q=0}^{\infty} P_{1,0;n_g,n_q} u_g^{n_g} u_q^{n_q} \tag{4}$$

$$Q(u_g, u_q; Y) = \sum_{n_g,n_q=0}^{\infty} P_{0,1;n_g,n_q} u_g^{n_g} u_q^{n_q} \tag{5}$$

We believe that each of m_g gluons and m_q quarks evolves independently, so the overall generating function looks like:

$$\sum_{n_g,n_q=0}^{\infty} P_{m_g,m_q;n_g,n_q} u_g^{n_g} u_q^{n_q} = [G(u_g,u_q;Y)]^{m_g}[Q(u_g,u_q;Y)]^{m_q} \tag{6}$$

Since the process is homogeneous in Y the transition probabilities obey Chapman-Kolmogorov equations, that's why for the gluon jet:

$$P_{1,0;n_g,n_q}(Y+Y') = \sum_{l_g,l_q=0}^{\infty} P_{1,0;l_g,l_q}(Y) P_{l_g,l_q;n_g,n_q}(Y') \tag{7}$$

Using equations (4) and (7) we obtain:

$$\sum_{n_g,nq=0}^{\infty} P_{1,0;n_g,n_q}(Y+Y') u_g^{n_g} u_q^{n_q} = G(u_g,u_q;Y+Y') =$$
$$\sum_{n_g,nq=0}^{\infty} \sum_{l_g,l_q=0}^{\infty} P_{1,0;l_g,l_q}(Y) P_{l_g,l_q;n_g,n_q}(Y') u_g^{n_g} u_q^{n_q} = [eq.(6)] =$$
$$\sum_{l_g,l_q=0}^{\infty} P_{1,0;l_g,l_q}(Y)[G(u_g,u_q;Y')]^{l_g}[Q(u_g,u_q;Y')]^{l_q} = G[G(u_g,u_q;Y'),Q(u_g,u_q;Y');Y] \tag{8}$$

Finally,:

$$G(u_g,u_q;Y+Y') = [G(u_g,u_q;Y'),Q(u_g,u_q;Y');Y] \tag{9}$$

Similarly, for quark jet:

$$P_{0,1;n_g,n_q}(Y+Y') = \sum_{l_g,l_q=0}^{\infty} P_{0,1;l_g,l_q}(Y) P_{l_g,l_q;n_g,n_q}(Y') \tag{10}$$

$$Q(u_g,u_q;Y+Y') = Q[G(u_g,u_q;Y'),Q(u_g,u_q;Y');Y] \tag{11}$$

It can be shown that [4]:

$$G(u_g,u_q;\Delta Y) = u_g + w^{(g)}(u_g,u_q)\Delta Y + o(\Delta Y) \tag{12}$$

$$Q(u_g,u_q;\Delta Y) = u_q + w^{(q)}(u_g,u_q)\Delta Y + o(\Delta Y) \tag{13}$$

In (12-13) equations the infinitesimal generating functions for quark $w^{(q)}(u_g, u_q)$ and gluon $w^{(g)}(u_g, u_q)$ jets are introduced:

$$w^{(g)}(u_g, u_q) = (-A - B)u_g + Au_g^2 + Bu_q^2 \quad (14)$$

$$w^{(q)}(u_g, u_q) = -\tilde{A}u_q + \tilde{A}u_q u_g \quad (15)$$

Using equations (9) and (12):

$$G(u_g, u_q; Y' + \Delta Y) = G[G(u_g, u_q; Y'), Q(u_g, u_q; Y'); \Delta Y] = \\ G(u_g, u_q; Y') + w^{(g)}(G(u_g, u_q; Y'), Q(u_g, u_q; Y'))\Delta Y + o(\Delta Y) \quad (16)$$

Using Taylor series expansion $f(x + \Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x$ and substituting $Y' \to Y$ we get backward Kolmogorov equation for gluon jet:

$$\frac{\partial G(u_g, u_q; Y)}{\partial Y} = w^{(g)}(G(u_g, u_q; Y), Q(u_g, u_q; Y)) \quad (17)$$

Simultaneously, for quark jet:

$$\frac{\partial Q(u_g, u_q; Y)}{\partial Y} = w^{(q)}(G(u_g, u_q; Y), Q(u_g, u_q; Y)) \quad (18)$$

The initial conditions are:

$$G(u_g, u_q; 0) = u_g \quad Q(u_g, u_q; 0) = u_q \quad (19)$$

Using (14) and (15) we get

$$\frac{\partial G}{\partial Y} = AG^2 + BQ^2 + (-A - B)G \quad (20)$$

$$\frac{\partial Q}{\partial Y} = -\tilde{A}G + \tilde{A}GQ \quad (21)$$

We can find the probability for quark (or gluon) to produce in the interval $(Y + \Delta Y)$ n_g gluons and n_q quarks through elementary process 1-3. It follows for quark jet:

$$P_{0,1;n_g,n_q}(Y + \Delta Y) = A(n_g - 1)P_{0,1;n_g-1,n_q}\Delta Y + \tilde{A}n_q P_{0,1;n_g-1,n_q}\Delta Y \\ + B(n_g + 1)P_{0,1;n_g+1,n_q-2}\Delta Y + [1 - An_g - \tilde{A}n_q - Bn_g]P_{0,1;n_g,n_q}\Delta Y \quad (22)$$

Dividing on ΔY and letting $\Delta Y \to 0$ we get the system of differential equations:

$$\frac{dP_{0,1;n_g,n_q}(Y)}{dY} = (-An_g - \tilde{A}n_q - Bn_g)P_{0,1;n_g,n_q} + A(n_g - 1)P_{0,1;n_g-1,n_q} + \\ \tilde{A}n_q P_{0,1;n_g-1,n_q} + B(n_g + 1)P_{0,1;n_g+1,n_q-2} \quad (23)$$

We neglect the quark-antiquark pair production, because it gives the smallest part in process, that is why$A \neq \tilde{A} \neq 0$, $B = 0$. In our work, we further will consider the quark jet, so from (23) we get:

$$\frac{dP_{0,1;n_g,1}}{dY} = -\tilde{A}P_{0,1;n_g,1} + \tilde{A}P_{0,1;n_g-1,1} - An_gP_{0,1;n_g,1} + A(n_g-1)P_{0,1;n_g-1,1} \quad (24)$$

Initial conditions:

$$P_{0,1;n_g,1}(0) = \begin{cases} 1, & n_g = 0 \\ 0, & n_g \neq 0 \end{cases} \quad (25)$$

To solve (24) system of equations we use the recurrent method:

$$n_g = 0 \quad \rightarrow \quad \frac{dP_0}{dY} = -\tilde{A}P_0 \quad (26)$$

$$P_{0,1;0,1} = P_0 = e^{-\tilde{A}Y} \quad (27)$$

$$n_g = 1 \quad \rightarrow \quad \frac{dP_1}{dY} = -\tilde{A}P_1 + \tilde{A}P_0 - AP_1 \quad (28)$$

$$P_{0,1;1,1} = P_1 = \frac{\tilde{A}}{A}e^{-\tilde{A}Y}(1 - e^{-AY}) \quad (29)$$

$$n_g = 2 \quad \rightarrow \quad \frac{dP_1}{dY} = -(\tilde{A} + 2A)P_2 + (\tilde{A} + A)P_1 \quad (30)$$

Let us introduce new parameter $k_p = \frac{\tilde{A}}{A}$ which shows the ratio of bremsstrahlung to gluon fission:

$$P_2 = \frac{k_p(k_p+1)}{2!}e^{-\tilde{A}Y}(1 - e^{-AY})^2 \quad (31)$$

Finally, multiplicity distribution in quark jet:

$$P_{n_g} = \frac{k_p(k_p+1)...(k_p+n_g-1)}{n_g!}e^{-\tilde{A}Y}(1 - e^{-AY})^{n_g} \quad (32)$$

$$\frac{k_p(k_p+1)...(k_p+n_g-1)}{n_g!} = \frac{(k_p+n_g-1)!}{n_g!(k_p-1)!} = C^{n_g}_{k_p+n_g-1} \quad (33)$$

And using the fact that $\tilde{A} = k_pA$:

$$P_{n_g} = C^{n_g}_{k_p+n_g-1}(e^{-AY})^{k_p}(1 - e^{-AY})^{n_g} \quad (34)$$

which is known as Negative Binomial Distribution.

The corresponding generating function can be found using the fact that:

$$P_m = \frac{1}{m!}\frac{\partial^m Q}{\partial u_g^m}\bigg|_{u_g=0} \tag{35}$$

$$Q(u_g) = u_q e^{-\tilde{A}Y}\left[1-(1-e^{-AY})u_g\right]^{-k_p} = u_q\left(\frac{e^{-AY}}{1-(1-e^{-AY})u_g}\right)^{k_p} \tag{36}$$

Let us find mean gluon number:

$$\overline{m} = \frac{\partial Q}{\partial u_g}\bigg|_{u_g=1} = k_p(e^{AY}-1) \tag{37}$$

$$e^{-AY} = \frac{k_p}{k_p+\overline{m}} \qquad e^{-\tilde{A}Y} = \left(\frac{k_p}{k_p+\overline{m}}\right)^{k_p} \tag{38}$$

Finally:

$$P_m = \frac{k_p(k_p+1)...(k_p+m-1)}{m!}\left(\frac{k_p}{k_p+\overline{m}}\right)^{k_p}\left(\frac{\overline{m}}{k_p+\overline{m}}\right)^m \tag{39}$$

The hadronization stage is described using a binomial distribution. This is based on the analysis of experimental data from e^+e^- annihilation [6]. At energies below 9 GeV the second correlation moment has negative value.

The hadronic multiparticle distribution:

$$P_n^H(z) = C_{N_p}^n\left(\frac{\overline{n}_p^h}{N_p}\right)^n\left(1-\frac{\overline{n}_p^h}{N_p}\right)^{N_p-n} \tag{40}$$

where $C_N^n = \frac{N!}{n!(N-n)!}$ - binomial coefficients, $p = q, g$ - type of parton, $\overline{n}_p^h$- the average hadron yield per parton, N_p- maximum secondaries of hadrons are formed from parton on the stage of hadronization.

The corresponding generation function:

$$Q_p^H = \left[1+\frac{\overline{n}_p^h}{N_p}(z-1)\right]^{N_p} \tag{41}$$

MD of hadrons in e^+e^--annihilation is determined by convolution of two stages (we consider 2 quarks and m gluons from which hadrons are formed):

$$P_n(s) = \sum_{m=0}^{\infty} P_m^P (Q^H)^{2+m}|_{z=0} \tag{42}$$

We assume that probabilities of formation of hadron from quark or gluon are equal:

$$\frac{\overline{n}_q^h}{N_q} = \frac{\overline{n}_g^h}{N_g} \tag{43}$$

We also introduce parameter $\alpha = N_g/N_q = \overline{n}_g^h/\overline{n}_q^h$ for distinguishing between hadron jets, created from quark or gluon on the second stage.

Letting $N = N_q,\ \overline{n}^h = \overline{n}_q^h$ we obtain:

$$Q_q^H(z) = \left[1 + \frac{\overline{n}^h}{N}(z-1)\right]^N \tag{44}$$

$$Q_g^H(z) = \left[1 + \frac{\overline{n}^h}{N}(z-1)\right]^{\alpha N} \tag{45}$$

$$P_n(s) = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\sum_{m=0}^{\infty} P_m^P[(Q_q^H)^2(Q_g^H)^m] \tag{46}$$

The multiparticle distribution of hadrons in e^+e^--annihilation in Two Stage Model:

$$P_n(s) = \sum_{m=0}^{\infty} P_m^P C_{(2+\alpha m)N}^m \left(\frac{\overline{n}^h}{N}\right)^n \left(1 - \frac{\overline{n}^h}{N}\right)^{(2+\alpha m)N-n} \tag{47}$$

This quantity $P_n(s)$ is the probability of obtaining a certain number n charged hadrons from m partons.

For comparing with experimental data the normalized factor Ω was introduced into (47) and a number of gluons in the sum was restricted by MG - maximal number of possible gluons created on the first stage

$$P_n(s) = \Omega\sum_{m=0}^{MG} P_m^P C_{(2+\alpha m)N}^m \left(\frac{\overline{n}^h}{N}\right)^n \left(1 - \frac{\overline{n}^h}{N}\right)^{(2+\alpha m)N-n} \tag{48}$$

5 Fitting

Multiplicity distribution P_n obtained in the experiments is the ratio of cross-sections $P_n = \sigma_n/\sigma$, where $\sigma = \sum_n \sigma_n$. We fitted experimental data for 14-43 GeV from TASSO collaboration [2] using the Minuit2 (for 14 and 22 GeV) Fumili2 (for 34. 8 and 43.6 GeV) minimization CERN ROOT package [3]. In our analysis the MG (maximal number of possible gluons created on the first stage) was equal to 20. The results

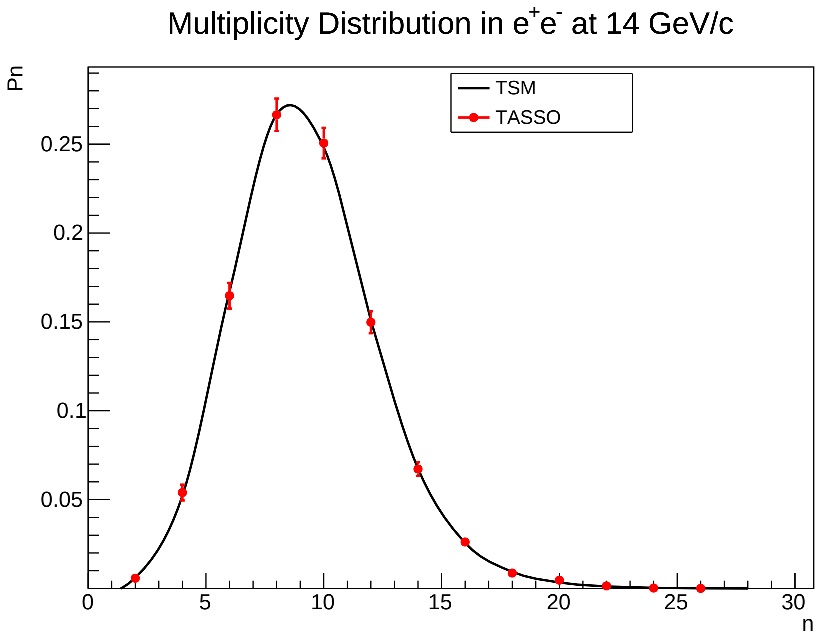

Figure 2: Multiplicity distribution for charged particles in e^+e^--annihilation for 14 GeV.

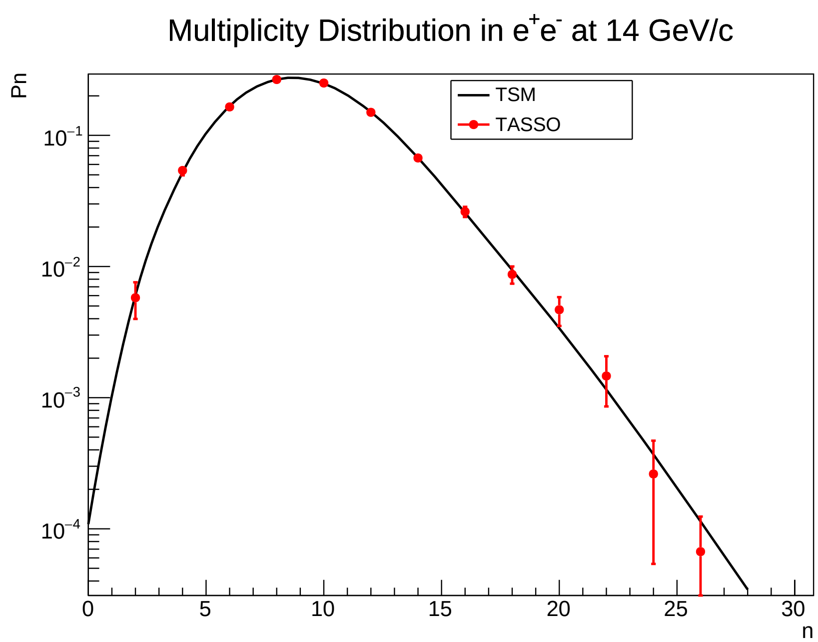

Figure 3: Multiplicity distribution for charged particles in e^+e^--annihilation for 14 GeV on a double logarithmic scale.

of comparison of model expression (48) with experimental data are represented in table 1 (parameters of TSM) and on figures 2-9.

Table 1 shows that the average multiplicity of gluons $\overline{m}$ has tendency to rise with energy. Parameter α was introduced for comparison of quark and gluon jets.

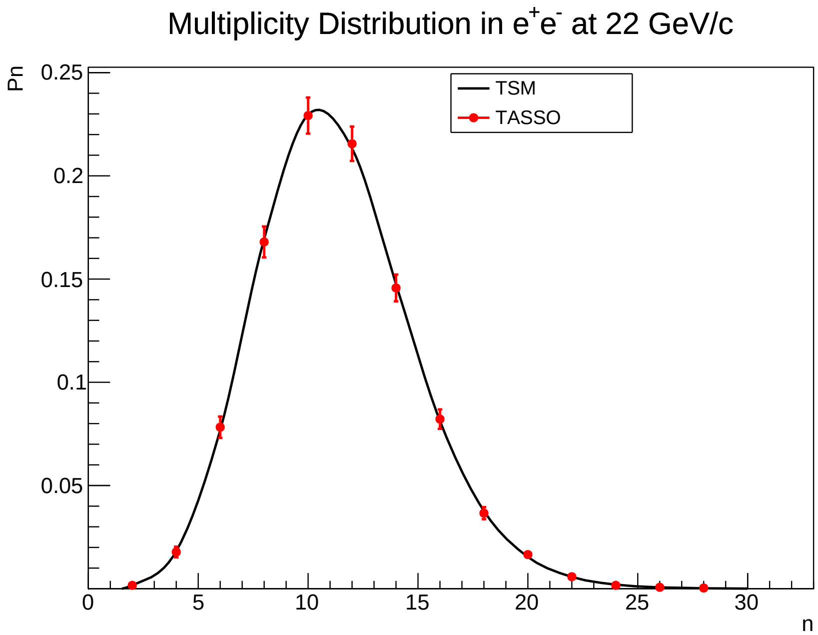

Figure 4: Multiplicity distribution for charged particles in e^+e^--annihilation for 22 GeV.

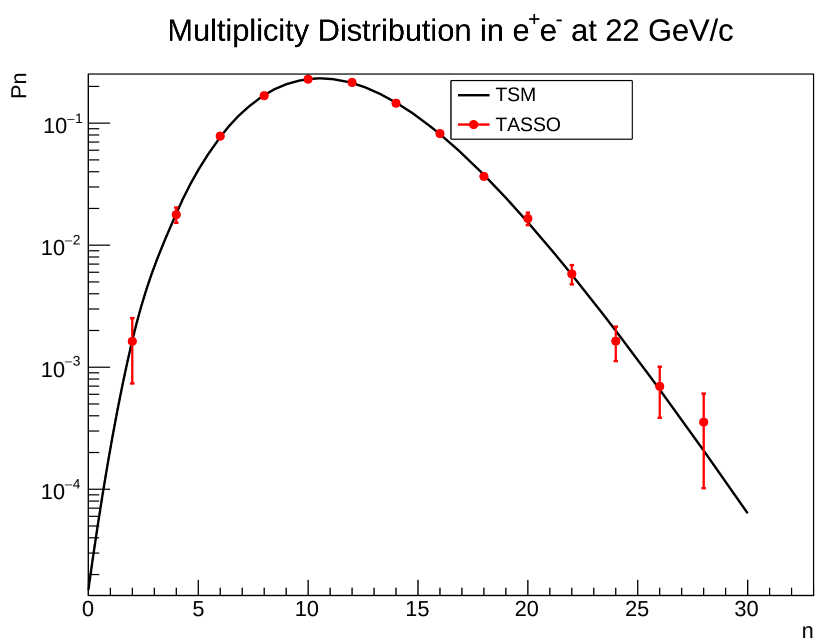

Figure 5: Multiplicity distribution for charged particles in e^+e^--annihilation for 22 GeV on a double logarithmic scale.

The fact that $\alpha < 1$ means that gluon jets are softer than quark one. This is consistent with the results obtained by Konishi, Ukawa and Veneziano [5].

The average number of hadrons formed from gluon is equal to $\overline{n}_g^h = \alpha \overline{n}_q^h$. The table 1 suggests that $\overline{n}_g^h \sim 1$ indicating the fragmentation mechanism of hadronization.

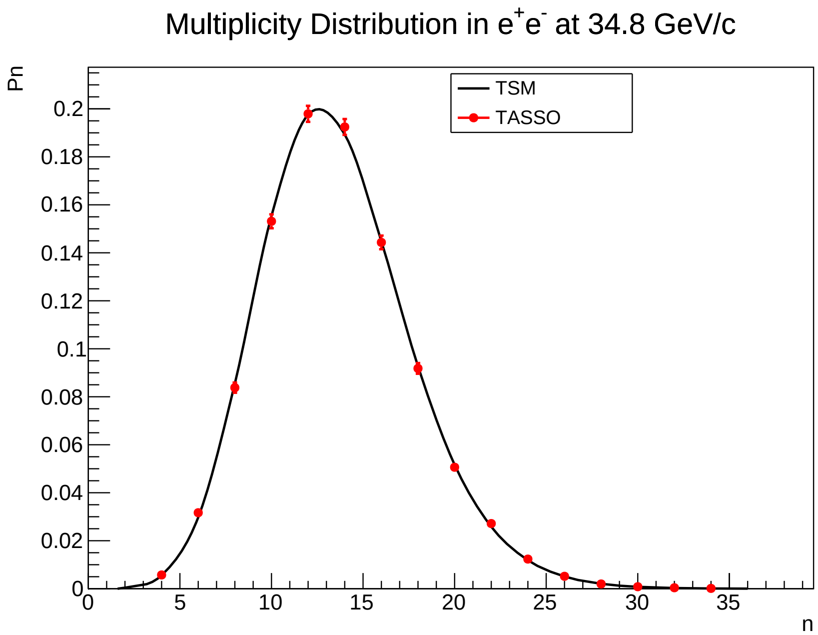

Figure 6: Multiplicity distribution for charged particles in e^+e^--annihilation for 34.8 GeV.

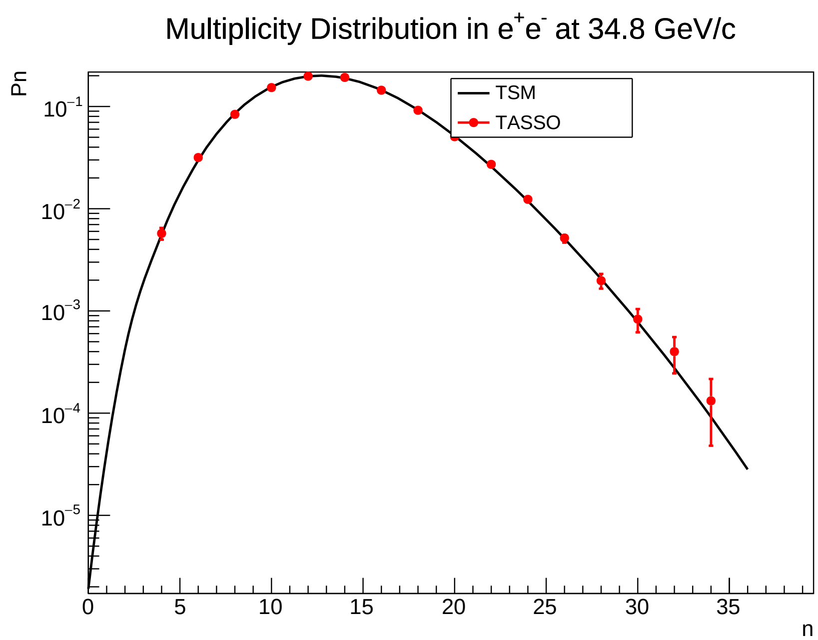

Figure 7: Multiplicity distribution for charged particles in e^+e^--annihilation for 34.8 GeV on a double logarithmic scale.

The parameter Ω must equal 2 due to satisfy charge conservation. Since the initial system is electrically neutral before the collision, it must also remain neutral after hadronization. This necessitates that all charged particles are produced in pairs.

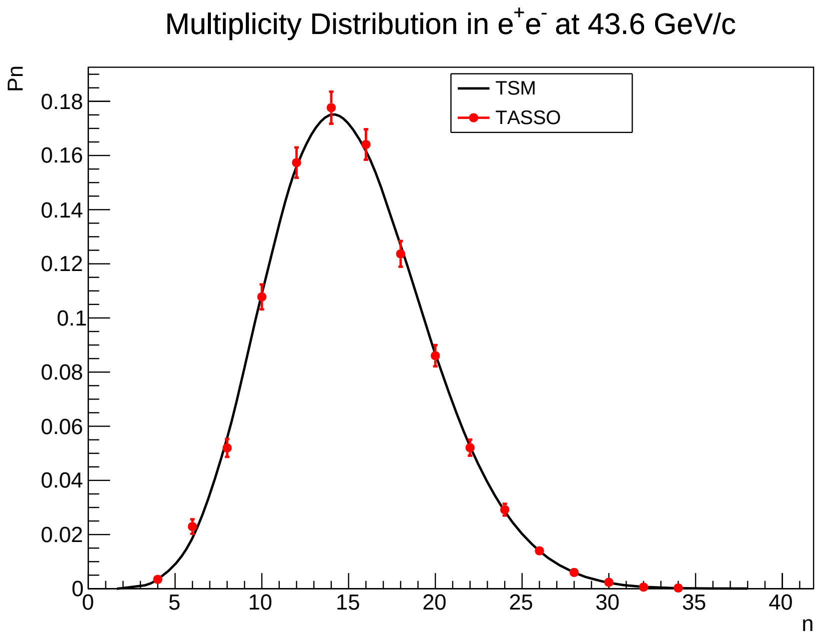

Figure 8: Multiplicity distribution for charged particles in e^+e^--annihilation for 43.6 GeV.

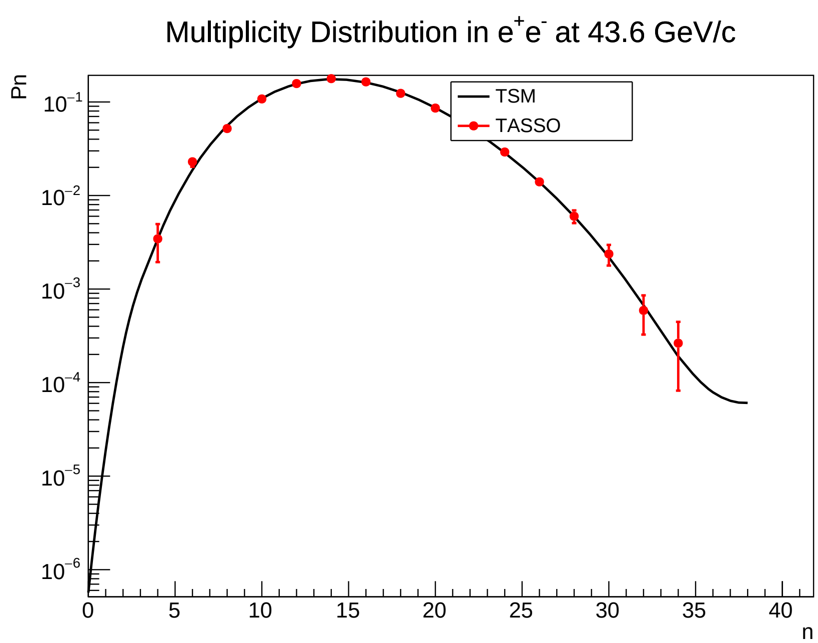

Figure 9: Multiplicity distribution for charged particles in e^+e^--annihilation for 43.6 GeV on a double logarithmic scale.

6 Three-gluon quarkonium decay

Let us consider three-gluon $\Upsilon(b\bar{b})$ decays:

$$e^+e^- \to q\bar{q} \to 3\, gluons \to 3\, jets$$

$\sqrt{s}, GeV$	k_p	$\overline{m}$	$\overline{n}_q^h$	N_q	α	Ω	χ^2
14	80.57	0.08	4.47	27.88	0.98	2.00	2.76
22	3.14	2.09	4.66	27.64	0.20	2.00	1.65
34.8	2.67	4.30	5.40	62.45	0.12	2.00	6.82
43.6	39.56	9.68	2.13	3.63	0.53	2.00	5.42

Table 1: Parameters of TSM

The generating function for the gluon jet which corresponds to the Furry-Yule distribution:

$$G(z) = \frac{z}{\overline{m}} \left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-1}, \tag{1}$$

where $\overline{m}$-average multiplicity for 1 gluon jet.

We assume that gluon jets evolve independently, therefore:

$$G^{(3g)}(z) = \left(\frac{z}{\overline{m}}\right)^3 \left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-3} \tag{2}$$

Let us find the first correlation moment using generating function (2):

$$f_1 = \overline{n} = \left.\frac{\partial G^{(3g)}(z)}{\partial z}\right|_{z=1} = \frac{3z^2}{\overline{m}^3}\left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-3} + \\ 3\left(\frac{z}{\overline{m}}\right)^3 \left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-4}\left(1 - \frac{1}{\overline{m}}\right) = 3 - 3\overline{m}^{-3}\overline{m}^4\left(1 - \frac{1}{\overline{m}}\right) = 3\overline{m} \tag{3}$$

The second correlation moment for generating function $G^{(3g)}(z)$ (2):

$$f_2 = G''(z) - (G'(z))^2 \tag{4}$$

First we calculate $G''(z)$:

$$G''(z) = \left.\frac{\partial^2 G^{(3g)}(z)}{\partial z^2}\right|_{z=1} = \frac{6z}{\overline{m}^3}\left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-3} + \\ \frac{3z^2}{\overline{m}^3}(-3)\left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-4}(-1)\left(1 - \frac{1}{\overline{m}}\right) + \frac{3z^2}{\overline{m}^3}(3)\left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-4}\left(1 - \frac{1}{\overline{m}}\right) + \\ 12\left(\frac{z}{\overline{m}}\right)^3\left[1 - z\left(1 - \frac{1}{\overline{m}}\right)\right]^{-5}\left(1 - \frac{1}{\overline{m}}\right)^2 = \\ 18\overline{m} + 12\overline{m}^2 - 24\overline{m} = 12\overline{m}^2 - 6\overline{m} = 6\overline{m}(2\overline{m} - 1) \tag{5}$$

Finally:

$$f_2 = 12\overline{m}^2 - 6\overline{m} - 9\overline{m}^2 = 3\overline{m}(\overline{m} - 2) \tag{6}$$

Let us express the second correlation moment in terms of the average multiplicity in case of three gluon jets:

$$f_2 = \overline{n}\left(\frac{\overline{n}}{3} - 2\right), \tag{7}$$

Multiplicity distribution function for $G^{(3g)}(z)$ can be found as:

$$P_n^{(3g)} = \frac{1}{n!}\frac{\partial^n}{\partial z^n} G^{(3g)}(s,z)|_{z=0} = \frac{1}{n!}\frac{\partial^n}{\partial z^n}\left(\frac{z}{\overline{n}/3}\right)^3 \left[1 - z(1 - \frac{1}{\overline{n}/3})\right]^{-3}|_{z=0} =$$
$$\frac{1}{(n-3)!}\left(\frac{1}{\overline{n}/3}\right)^3 \frac{\partial^{n-3}}{\partial z^{n-3}}\left[1 - z(1 - \frac{1}{\overline{n}/3})\right]^{-3}|_{z=0} \tag{8}$$

$$P_n^{(3g)} = \frac{(-3)(-4)...(-3-n+2)}{(n-3)!}(-1)^{n-3}\left(\frac{1}{\overline{n}/3}\right)^3 \left(1 - \frac{1}{\overline{n}/3}\right)^{n-3} \tag{9}$$

$$P_n^{(3g)} = \frac{(n-2)(n-1)}{2(\overline{n}/3)^3}(1 - \frac{1}{\overline{n}/3})^{n-3} \tag{10}$$

From (10) it follows that the MD differs from zero starting from $n = 3$, i.e., when there are not fewer than three gluons in the cascade. In this case it is possible to introduce a new variable $n' = n - 3$:

$$P_{n'}^{(3g)} = \frac{(n'+2)(n'+1)}{2(\overline{n}/3)^3}\left(1 - \frac{1}{\overline{n}/3}\right)^{n'} \tag{11}$$

The generation function for hadronization stage is: (binomial distribution):

$$Q^H(z) = \left(1 + \frac{\overline{n}_g^h}{N_g}(z-1)\right) \tag{12}$$

The hadronic multiparticle distribution:

$$P_k^H(z) = C_{N_g}^k \left(\frac{\overline{n}_g^h}{N_g}\right)^k \left(1 - \frac{\overline{n}_g^h}{N_g}\right)^{N_g - k} \tag{13}$$

where $C_{N_g}^k = (N_g!)/(k!(N_g-k)!)$ - binomial coefficients, $\overline{n}_g^h$- the average hadron yield per gluon, N_g- maximum secondaries of hadrons are formed from gluon on the stage of hadronization.

The convolution of two stages - the cascade and hadronization - gives us the final distribution of the number of hadrons in the three-gluon decay of quarkonia:

$$P_k = \sum_{n'=0} P_{n'} C^k_{(3+n')Ng} \left(\frac{\overline{n}^h_g}{N_g}\right)^k \left(1 - \frac{\overline{n}^h_g}{N_g}\right)^{(3+n')N_g - k} \tag{14}$$

Average multiplicity for upsilon (bottomonium) decays to three gluons with hadronization:

$$f_1 = \overline{k} = \sum_{k=0}^{N_g} P_k \cdot k = \sum_{k=0}^{N_g} \sum_{n'=0} P_{n'} \frac{((3+n')N_g)! \cdot k}{k!((3+n')N_g - k)!} \left(\frac{\overline{n}^h_g}{N_g}\right)^k \left(1 - \frac{\overline{n}^h_g}{N_g}\right)^{(3+n')N_g - k} =$$
$$\sum_{n'=0} P_{n'} \cdot (3+n') N_g \cdot \frac{\overline{n}^h_g}{N_g} \sum_{k=1}^{N_g} \frac{((3+n')N_g - 1)!}{(k-1)!((3+n')N_g - k)!} \left(\frac{\overline{n}^h_g}{N_g}\right)^{k-1} \left(1 - \frac{\overline{n}^h_g}{N_g}\right)^{(3+n')N_g - k} =$$
$$\overline{n}^h_g \sum_{n'=0} P_{n'} \cdot (3 + n') = \overline{n}^h_g (3 + \overline{n'}) = \overline{n}^h_g \overline{n} \tag{15}$$

The second correlation moment for the distribution of the number of hadrons in the three-gluon decay of quarkonia

$$f_2 = \overline{k^2} - (\overline{k})^2 - \overline{k} \tag{16}$$

First, we calculate:

$$\overline{k^2} = \sum_{k=0}^{N_g} P_k \cdot k^2 \tag{17}$$

$$\overline{k^2} = \sum_{k=0}^{N_g} \sum_{n'=0} P_{n'} \frac{((3+n')N_g)! \cdot k \cdot (k-1+1)}{k!((3+n')N_g - k)!} \left(\frac{\overline{n}^h_g}{N_g}\right)^k \left(1 - \frac{\overline{n}^h_g}{N_g}\right)^{(3+n')N_g - k} =$$
$$\sum_{n'=0} P_{n'} (3+n') \overline{n}^h_g \times$$
$$\left[1 + ((3+n')\overline{n}^h_g - \left(\frac{\overline{n}^h_g}{N_g}\right)) \sum_{k=1}^{N_g} \frac{((3+n')N_g - 2)!}{(k-2)!((3+n')N_g - k)!} \left(\frac{\overline{n}^h_g}{N_g}\right)^{k-2} \left(1 - \frac{\overline{n}^h_g}{N_g}\right)^{(3+n')N_g - k}\right]$$
$$= \sum_{n'=0} P_{n'} (3+n') \overline{n}^h_g + \sum_{n'=0} P_{n'} (3+n')^2 (\overline{n}^h_g)^2 - \sum_{n'=0} P_{n'} (3+n') \frac{(\overline{n}^h_g)^2}{N_g} =$$
$$\overline{n} \cdot \overline{n}^h_g + \overline{n^2} \cdot (\overline{n}^h_g)^2 - \overline{n} \cdot \frac{(\overline{n}^h_g)^2}{N_g} \tag{18}$$

Using equation (7) we obtain:

$$\overline{n^2} = f_2 + \overline{n} + (\overline{n})^2 = \overline{n}\left(\frac{\overline{n}}{3} - 2\right) + \overline{n} + (\overline{n})^2 = \frac{4}{3}(\overline{n})^2 - \overline{n} \tag{19}$$

Inserting (19) in (18) we get:

$$\overline{k^2} = \overline{n} \cdot \overline{n}_g^h + (\frac{4}{3}(\overline{n})^2 - \overline{n}) \cdot (\overline{n}_g^h)^2 - \overline{n} \cdot \frac{(\overline{n}_g^h)^2}{N_g} = \overline{k} + \frac{4}{3}(\overline{k})^2 - \frac{(\overline{k})^2}{\overline{n}} - \frac{(\overline{k})^2}{\overline{n}N_g} \tag{20}$$

The second correlation moment for upsilon (bottomonium) decays to three gluons with hadronization:

$$f_2 = \overline{k^2} - (\overline{k})^2 - \overline{k} = \frac{1}{3}(\overline{k})^2 - \frac{(\overline{k})^2}{\overline{n}} - \frac{(\overline{k})^2}{\overline{n}N_g} = (\overline{k})^2\left(\frac{1}{3} - \frac{1}{\overline{n}} - \frac{1}{\overline{n}N_g}\right) \tag{21}$$

It also can be written as following:

$$f_2 = 3\overline{m}(\overline{n}^h)^2\left(\overline{m} - 1 - \frac{1}{N_g}\right) \tag{22}$$

where $\overline{m}$ - average number of gluons that are formed in the cascade stage, N_g - maximum number of hadrons formed from a gluon in the hadronization stage, $\overline{n}^h$ - average number of these hadrons.

7 Conclusions

It has been shown that the multiplicity distributions obtained within the framework of the Two Stage model describe the behaviour of experimental data. The model parameters for 14, 22, 34.8, 43.6 GeV energies were found. Expressions for the second correlation moment and average multiplicity were obtained for the three-gluon decay of quarkonium.

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