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FINAL REPORT ON THE INTEREST PROGRAMME

Equilibrium distribution of heavy quarks inside a thermal quark-gluon plasma

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Contents

1 Abstract

The evolution of heavy quarks after high-energy collisions is an active area of research for understanding the behavior of the quark-gluon matter at extremely high temperatures and densities. One method to comprehend the numerous aspects of QGP is to probe the plasma medium with high-energy particles that are created immediately after the collision before the medium is actually formed. This report aims to test the equilibrium distribution features through the Fokker-Planck equation to asses whether it undergoes Boltzmann-Gibbs statistics or Tsallis statistics.

Key words: Equilibrium distribution, Fokker-Planck Equation, Tsallis statistics.

2 Introduction

Objects around us are made up of matter particles (Fermions) and mediator particles (Bosons). They interact together via four interactions: electromagnetic interaction, gravitational interaction, weak interaction, and strong interaction. Figure.1 classifies the three generations of Fermions and Bosons within their mass, charge, spin, and name. A Quark is a fundamental particle that contains color charges (red, green, and blue) that were introduced to comply with Pauli's exclusion principle. Under extreme conditions (high temperature and/or density), hadrons (e.g. protons and neutrons), the composite particles made up of quarks and gluons, lose their identity and form a plasma of quarks and gluons, termed as Quark Gluon Plasma (QGP). This is a state of matter in which the micro-second old universe existed. Studying the transport of heavy quarks through the QGP medium is one of the ways to study the properties of QGP and in this project, we focus on this phenomenon. Heavy quarks (charm, bottom, and top) are characterized by larger mass (m_Q) compared to those of the light quarks (m_q), i.e. $m_Q >> m_q$.. Their Compton wavelength $\lambda_Q << R_{had}$ where R_{had} is the size of the hadron containing the heavy quark, $\lambda_Q \sim \frac{1}{m_Q}$ and $R_{\mathrm{had}} \sim m_q$ [1].

- Heavy quarks including charm and bottom quarks are important in the studies of QGP because of the following reasons:

  - The production of HQs is essentially confined to the initial, phase of a heavy-ion collision and they do not control the bulk properties of the matter. HQ mass is considerably larger than the generally acquired temperatures and other nonperturbative measurements, so the system's full space-time evolution has been recorded by the heavy flavors.
  - Heavy quarks can successfully maintain the interaction history because their thermalization time scale is longer than that of light quarks and gluons by a factor of $\frac{m_Q}{T}$ [2].

The project aims to study the equilibrium distribution of the heavy quarks after they complete their passage through the QGP medium with the help of the Fokker-Planck equation. It is organized as follows; we first review what a QGP is, its importance, and the passage of HQs inside the QGP. Then we begin to study the equilibrium distribution of HQs with the aid of evolution equations;

Boltzmann Transport Equation (BTE), and the Fokker-Planck Equation (FPE) through certain approximated approaches. Finally, model the distribution using MATHEMATICA to analyze the dependency on the transport coefficients, to test whether this equilibrium obeys Boltzmann or Tsallis statistics.
In this project, the natural units are used where $\hbar = c = k_B = 1$, as well as the Einstein summation convention.

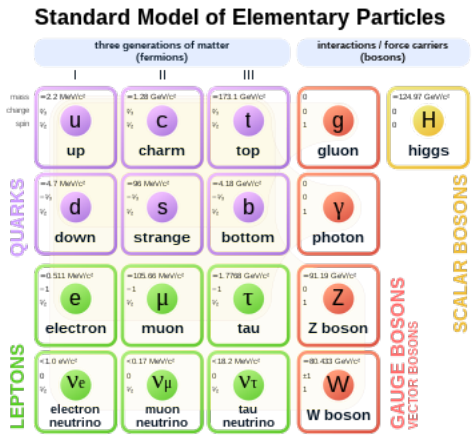

Figure 1: Fundamental particles classification [3].

3 Quark Gluon Plasma

3.1 QGP formation

QGP is a thermalized state beyond $T_c = 170\text{MeV}$ and $\epsilon = 1\text{GeV/fm}^3$ in which quarks and gluons are liberated to move within a nuclear volume which displays the screening of color charges. As shown in Fig.(2), high density/ temperature associated with the increment in energy results in the creation of Quark-Gluon Plasma (QGP), in which the quarks are then deconfined. Deconfinement is the phenomenon where quarks can not be bound into their hadrons due to screening of color potential given by $V(r) \sim \frac{-\alpha}{r} + \sigma r$.[1]

3.2 Importance of QGP

At the early stages of universe evolution, as it expands, strong and weak interactions are detached and it was a deconfined state of quarks and gluons $T \sim 100GeV$ known as QGP. Typically, at $T \sim 100MeV$ hadrons are created and so occurs the deconfinement-confinement transition. QGP might also be present at the center of a neutron star since it has a high central density (~ 10 normal nuclear matter density), thus hadrons lose their identities and take the form of QGP. The temperature is a key distinction between QGP in the early cosmos and that in neutron stars. Whereas QGP has a temperature of $T \sim 100MeV$ in the early cosmos, it has a temperature of $T \sim 0MeV$ at the core of a neutron star. [1]

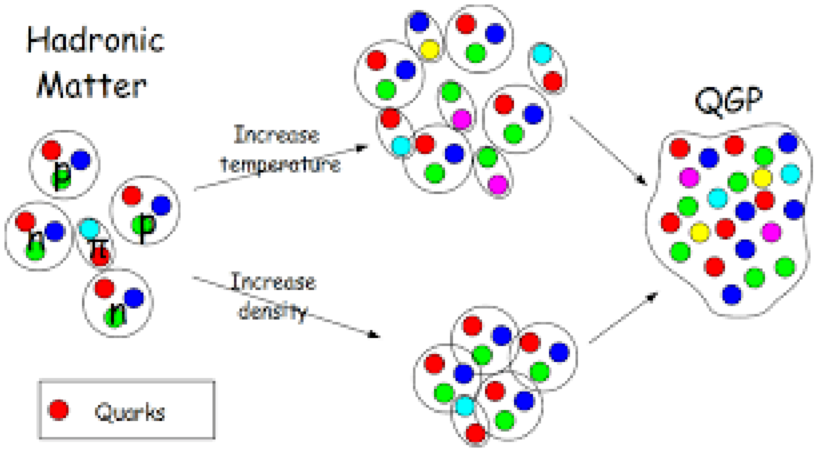

Figure 2: QGP in high density/temperature [4]

3.3 The dynamics of HQs inside the QGP

Within 1fm after two lead ion beams collide, the hydrodynamic expansion of a thermalized medium begins. [5], [6]. Fig.(3) visualizes the evolution of high energy particles inside the QGP medium where the collision point of Pb+Pb is at the origin point. After the collision, there is a pre-equilibrium phase where high energy particles are liberated until reaching a proper time τ_o less than 1 fm. During the phase equilibrium, a nearly thermal quark-gluon plasma is formed. Hadrons are believed to be split by a first-order phase transition and form an expanded QGP medium. The plasma thermalizes at a temperature of around $2T_c$; when it cools, it mostly expands longitudinally until it reaches a temperature below T_c, at which point it transforms into a gas of hadrons. The elemental composition of the created hadrons freezes at that temperature, causing the phase shift from the QGP phase to the hadronic phase [5], [6].

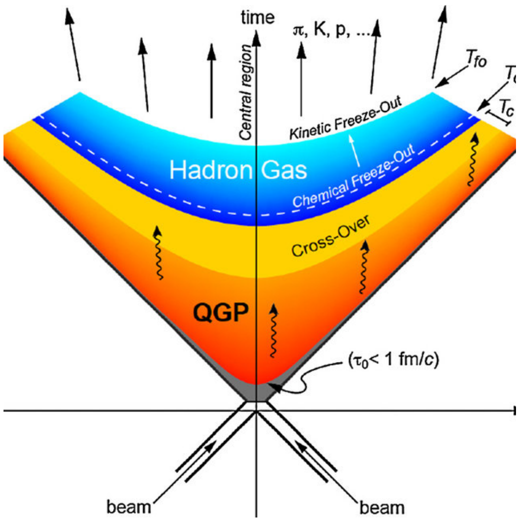

Figure 3: Space-time diagram of evolution after high-energy collisions [7].

4 Evolution equations

One may encounter two types of evolution while studying QGP,

1. Hydrodynamic evolution of medium governed by $\partial_\mu T^{\mu\nu} = 0$ where $T^{\mu\nu}$ is the Stress Energy Tensor[8].

2. Evolution of energetic particles like HQ having non-equilibrium distribution

4.1 Boltzmann Transport Equation in the Relaxation Time Approximation

The evolution equation of HQs is given by the Boltzmann Transport Equation (BTE):

$$\frac{df}{dt} = C[f] \tag{1}$$

where f is the phase-space distribution function, and $C[f]$ is the collision term. In the relaxation time approximation (RTA) (for a homogeneous medium with no external force) :

$$C[f] \approx \frac{\partial f}{\partial t} = -\frac{f - f_{\text{eq}}}{\tau} \tag{2}$$

We note that τ or the relaxation time is the time taken to reach equilibrium. f_{eq} is the distribution function, meaning it is a constant quantity in time ($\frac{\partial f_{\text{eq}}}{\partial t}$=0). The minus sign indicates that collisions help the distribution to reach equilibrium [9]. Solving for $f(t)$;

$$\frac{\partial f}{f - f_{\text{eq}}} = -\frac{\partial t'}{\tau} \tag{3}$$

$$\int_{f(t=0)}^{f(t)} \frac{\partial f}{f - f_{\text{eq}}} = -\int_{t=0}^{t} \frac{\partial t'}{\tau}$$

$$\ln[f - f_{\text{eq}}]_{f(0)}^{f(t)} = -\frac{[t']_0^t}{\tau}$$

$$\ln\left(\frac{f(t) - f_{\text{eq}}}{f(0) - f_{\text{eq}}}\right) = -\frac{t}{\tau}$$

Defining $f(0) = f_{\text{in}}$, we obtain:

$$f(t) - f_{\text{eq}} = (f_{\text{in}} - f_{\text{eq}})e^{-\frac{t}{\tau}}$$

$$f(t) = f_{\text{eq}} + (f_{\text{in}} - f_{\text{eq}})e^{-\frac{t}{\tau}} \tag{4}$$

4.2 Deriving Fokker Planck Equation from Boltzmann Transport Equation

The distribution function $f(\vec{x}, \vec{p}, t)$, can be used to express the statistical features of a group of particles. The probability of finding the particle in an infinitesimally small area of n-dimensional phase space is calculated by multiplying this density by the phase-space volume element $d^n x d^n p$. Under the assumption of $f(\vec{x}, \vec{p}, t)$ obeying Boltzmann-Vlasov master equation, the collision term $W(\vec{p}, \vec{k})$ is absolutely local and only depends on the particle momenta. Where $\vec{p}$ is the 3-momentum of incoming particle, and $\vec{k}$ is the 3-momentum transfer. The collision term consists of two components; the first gain term $W(\vec{p}+\vec{k},\vec{k})$ indicates the rate at which a particle with momentum $\vec{p}+\vec{k}$ loses momentum $\vec{k}$ as a result of reactions with the medium. While the second loss term $W(\vec{p},\vec{k})$ is a result of scattering. The Fokker-Planck equation is then obtained by Taylor expanding the collision term about $\vec{p}$ in the second-order of $\vec{k}$ assuming soft collisions $|\vec{k}| \approx 0$ [9]

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} \\
&\quad = \int d^3k \left[W(\vec{p}+\vec{k},\vec{k}) f(\vec{x},\vec{p}+\vec{k},t) - W(\vec{p},\vec{k}) f(\vec{x},\vec{p},t) \right] \\
&\quad = \int d^3k \Big[W(\vec{p},\vec{k}) f(\vec{x},\vec{p},t) + \vec{k}\cdot\nabla_{\vec{p}} W(\vec{p},\vec{k}) f(\vec{x},\vec{p},t) \\
&\qquad\qquad + \frac{1}{2}\sum_{i,j} k_i k_j \frac{\partial^2 f}{\partial p_i \partial p_j} W(\vec{p},\vec{k}) - W(\vec{p},\vec{k}) f(\vec{x},\vec{p},t) \Big] \\
&\quad = \int d^3k \left[k_i \frac{\partial W}{\partial p_i} f(\vec{x},\vec{p},t) + \frac{1}{2}\sum_{i,j} k_i k_j \frac{\partial^2 f}{\partial p_i \partial p_j} W(\vec{p},\vec{k}) \right] \\
&\quad = \frac{\partial}{\partial p_i} A_i f + \frac{\partial^2}{\partial p_i \partial p_j} B_{ij} f
\end{aligned} \tag{5}
$$

Where A_i and B_{ij} are the drag and diffusion coefficients respectively, given by:

$$A_i(\vec{p}) = \int d^3k k_i W(\vec{p},\vec{k}) \tag{6}$$

$$B_{ij}(\vec{p}) = \int d^3k k_i k_j W(\vec{p},\vec{k}) \tag{7}$$

4.3 Fokker Planck Equation and the equilibrium function

Fokker Planck equation studies the phase space distribution of particles in a thermal medium. It can be obtained by applying the Landau Kinetic Approximation (LKA) to the BTE. In this approximation, the BTE can be written in terms of the drag and diffusion coefficients (A_i and B_{ij}) in the following way: [9], [10]

$$\frac{\partial f}{\partial t} + \dot{x}_i \frac{\partial f}{\partial x_i} + \dot{p}_i \frac{\partial f}{\partial p_i} = \frac{\partial}{\partial p_i} A_i f + \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} B_{ij} f \tag{8}$$

The stationary solution of this equation can be written as:

$$f_{\text{eq}}(p) = N \exp[-\phi(p)] \tag{9}$$

Assuming a homogeneous medium with no external force;

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial P_i}[A_i f + \frac{\partial B_{ij}}{\partial P_j} f] = -\vec{\nabla}_P \cdot \vec{\mathcal{P}} \tag{10}$$

Where $\vec{\mathcal{P}}$ is the Probability Current;

$$\mathcal{P}_i = -A_i f - \frac{\partial}{\partial P_j}(B_{ij} f) \tag{11}$$

For a stationary state i.e. at equilibrium where $\frac{\partial f}{\partial t} = 0$, the Probability current vanishes, therefore eq(10) becomes:

$$0 = -A_i N \exp[-\phi(p)] - \frac{\partial}{\partial P_j}(B_{ij} N \exp[-\phi(p)]) \tag{12}$$

$$-A_i(N \exp[-\phi(p)]) = \frac{\partial(B_{ij})}{\partial P_j}(N \exp[-\phi(p)]) + B_{ij}(N \exp[-\phi(p)])(-\frac{\partial \phi}{\partial P_j}) \tag{13}$$

Finally, we will be left with the drag coefficient;

$$\boxed{A_i(\vec{p}, T) = B_{ij}(\vec{p}, T)\frac{\partial \phi(\vec{p})}{\partial P_j} - \frac{\partial B_{ij}(\vec{p}, T)}{\partial P_j}} \tag{14}$$

Considering Einstein's summation convention, the equivalence of eq(14) in the case of spatial homogeneity where the equilibrium distribution related to the drag and diffusion coefficients depends on $p = |\vec{p}|$ will be derived in the following steps; The most general form of A_i and B_{ij} is given in terms of p_i (3 momenta of HQ) and δ_{ij} (Kronecker delta) as shown in eq(15) and eq(16):

$$A_i(\vec{p}, T) = p_i A(p, T) \tag{15}$$

$$B_{ij}(\vec{p}, T) = (\delta_{ij} - \frac{p_i p_j}{p^2})B_{\parallel}(\vec{p}, T) + \frac{p_i p_j}{p^2}B_{\perp}(\vec{p}, T) \tag{16}$$

Substituting eq(15) and eq(16) into eq(14):

$$\begin{aligned} A_i(p, T) &= B_{ij}\frac{\partial \phi(\vec{p})}{\partial p_j} - \frac{\partial B_{ij}(\vec{p}, T)}{\partial p_j} \\ &= p_i A(p, T) \end{aligned} \tag{17}$$

$$A(p, T) = \left[(\delta_{ij} - \frac{p_i p_j}{p^2})B_{\perp}\frac{\partial \phi(\vec{p})}{\partial p_j} + \frac{p_i p_j}{p^2}B_{\parallel}\frac{\partial \phi(\vec{p})}{\partial p_j} - \frac{\partial B_{ij}(\vec{p}, T)}{\partial p_j}\right]\frac{1}{p_i} \tag{18}$$

The first term contributes to zero when p^2 is written in form of $p_j p_j$:

$$\left[(\delta_{ij} - \frac{p_i p_j}{p^2})B_{\perp}\frac{\partial \phi(\vec{p})}{\partial p_j}\right]\frac{1}{p_i} = B_{\perp}\frac{\partial \phi(\vec{p})}{\partial p_j}\left[\frac{1}{p_j} - \frac{1}{p_j}\right] = 0 \tag{19}$$

Subsequently, the second term contributes to the following:

$$\left[\frac{p_i p_j}{p^2}B_{\parallel}\frac{\partial \phi}{\partial p_j}\right]\frac{1}{p_i} = \frac{1}{p}B_{\parallel}\frac{\partial \phi}{\partial p} \tag{20}$$

As for the third term $\frac{\partial B_{ij}}{\partial p_j}\frac{1}{p_i}$:

$$\begin{aligned} &\left[\delta_{ij}\frac{\partial B_{\perp}}{\partial p_j} + \frac{\partial}{\partial p_j}\left[\frac{p_i p_j}{p^2}\right][B_{\parallel} - B_{\perp}] + \left[\frac{p_i p_j}{p^2}\right]\left[\frac{\partial}{\partial p_j}\right][B_{\parallel} - B_{\perp}]\right]\frac{1}{p_i} \\ &= \frac{1}{p}\frac{dB_{\perp}}{dp} + \frac{1}{p_i}\frac{\partial}{\partial p_j}(\frac{p_i p_j}{p^2})[B_{\parallel} - B_{\perp}] + \frac{1}{p}\frac{d}{dp}[B_{\parallel} - B_{\perp}] \\ &= \frac{1}{p}\frac{dB_{\parallel}}{dp} + \frac{1}{p_i}\left[\frac{\partial}{\partial p_j}(\frac{p_i p_j}{p^2})\right][B_{\parallel} - B_{\perp}] \end{aligned} \tag{21}$$

'1q

Also, $\frac{\partial}{\partial p_j}(\frac{p_i p_j}{p^2})$ is given by:

$$
\begin{aligned}
&= \frac{1}{p^2}\frac{\partial}{\partial p_j}(p_i p_j) + p_i p_j \frac{\partial}{\partial p_j}\left[\frac{1}{p^2}\right] \\
&= \frac{1}{p^2}\left[\frac{\partial p_i}{\partial p_j}p_j + p_i\frac{\partial p_j}{\partial p_j}\right] - p_i p_j (p^2)^{-2}\left[\frac{\partial p^2}{\partial p_j}\right] \\
&= \frac{1}{p^2}(\delta_{ij}p_j + p_i\delta_{jj}) - p_i p_j (p^2)^{-2}\left[\frac{\partial p^2}{\partial p_j}\right] \\
&= \frac{1}{p^2}(p_i + 3p_i) - p_i p_j \frac{1}{p^4}\left[\frac{\partial p^2}{\partial p_j}\right]
\end{aligned}
\tag{22}
$$

The term $\frac{\partial p^2}{\partial p_j}$ can be evaluated to give the following by letting $p^2 = p_k p_k$ from Einstein's summation convention:

$$
\frac{\partial p^2}{\partial p_j} = 2\frac{\partial p_k}{\partial p_j}p_k = 2\delta_{kj}p_k = 2p_j \tag{23}
$$

Putting it back to eq(22), we will have:

$$
\frac{\partial}{\partial p_j}(\frac{p_i p_j}{p^2}) = \frac{1}{p^2}(4p_i) - \frac{2}{p^4}p_i p_j p_j \tag{24}
$$

Therefore, eq(21) will be:

$$
\begin{aligned}
&= \frac{1}{p}\frac{dB_\parallel}{dp} + \frac{1}{p_i}\left[\frac{1}{p^2}(4p_i) - \frac{2}{p^4}p_i p_j p_j\right][B_\parallel - B_\perp] \\
&= \frac{1}{p}\frac{dB_\parallel}{dp} + \frac{4}{p^2} - \frac{2}{p^4}p_j p_j[B_\parallel - B_\perp] \\
&= \frac{1}{p}\frac{dB_\parallel}{dp} + \left[\frac{2}{p^2}\right][B_\parallel - B_\perp]
\end{aligned}
\tag{25}
$$

Now by summing over the RHS of equations (20) and (25) in eq(18), we will have the drag coefficient in the following equation:

$$
\boxed{A(p,T) = \frac{1}{p}\frac{d\phi}{dp}B_\parallel - \frac{1}{p}\frac{dB_\parallel}{dp} - \frac{2}{p^2}[B_\parallel - B_\perp]} \tag{26}
$$

where $\delta_{ii} = 3$ in our case of 3D, and in nD the eq will be:

$$
\boxed{A(p,T) = \frac{1}{p}\frac{d\phi}{dp}B_\parallel - \frac{1}{p}\frac{dB_\parallel}{dp} - \frac{n-1}{p^2}[B_\parallel - B_\perp]} \tag{27}
$$

Now we choose the following form for ϕ :

$$
\phi_{Ts} = \frac{(1)}{1-q}\ln\left[1 - (1-q)\frac{E(p)}{T}\right] \tag{28}
$$

$$
\frac{d\phi_{Ts}}{dp} = \frac{\frac{1}{T}\frac{dE}{dp}}{1 - (1-q)\frac{E}{T}} \tag{29}
$$

From eq(27):

$$\frac{d\phi}{dp} = \frac{pA(p,T)}{B_{\parallel}} + \frac{1}{B_{\parallel}}\frac{dB_{\parallel}}{dp} + \frac{n-1}{p}\frac{1}{B_{\parallel}}[B_{\parallel} - B_{\perp}] \tag{30}$$

By equating the RHS of eq(27) and eq(29);

$$\frac{\frac{1}{T}\frac{dE}{dp}}{1-(1-q)\frac{E}{T}} = \frac{1}{B_{\parallel}}\left[pA + \frac{dB_{\parallel}}{dp} + \frac{n-1}{p}[B_{\parallel} - B_{\perp}]\right] \tag{31}$$

$$\boxed{[T+(q-1)E] = \frac{dE}{dp}\frac{B_{\parallel}}{pA + \frac{dB_{\parallel}}{dp} + \frac{n-1}{p}[B_{\parallel} - B_{\perp}]}} \tag{32}$$

4.4 Analysis of the equilibrium function

The explanation of the term ϕ_{Ts} in eq(28) is linked to Boltzmann Gibbs (BG) and Tsallis statistics where both theories approach large collections of particles. However, Tsalli statistics -parametrized by q, and T_{T}- is a generalization of the Boltzmann Gibbs statistics [9], [11].

- Boltzmann Gibbs Entropy is given by:

$$S = -k\Sigma p_i \ln p_i$$

  where k is a positive constant is equal to 1 according to the convention of natural units, and p_i is the probability related to the event

- Tsalli Entropy which depends on the parameter q:

$$S_{\mathrm{q}} = \frac{k}{1-q}\Sigma p_i(1-p_i^{q-1})$$

Boltzmann statistics is recovered when $q \to 1$ [9].
A mathematical explanation of the deviation between BG and Tsallis statistics is as follows:
Starting from $\phi_{Ts} = \frac{1}{1-q}\left[\ln[1-(1-q)\frac{E(p)}{T}\right]$:
let $x = -(1-q)\frac{E}{T}$ and Taylor expand $\ln(1+x)$ around $x = 0$

$$\ln(1+x) = \Sigma_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} \tag{33}$$

Subsequently,

$$\begin{aligned}\lim_{q\to 1}\frac{1}{1-q}\ln\left[1-(1-q)\frac{E}{T}\right] &= \left[\frac{1}{1-q}\left[-(1-q)\frac{E}{T}\right] + \frac{1}{2}\left[(1-q)\frac{E}{T}\right]^2\right.\\ &\quad \left. - \frac{1}{3}\left[(1-q)\frac{E}{T}\right]^3 + ...\right] \\ &= \frac{-E}{T}\end{aligned} \tag{34}$$

Therefore, Tsallis statistics is reduced giving BG equilibrium distribution function $\phi = \exp \frac{-E}{T_{\text{B}}}$.
The Boltzmann-Jüttner distribution $[q = 1, T_{\text{T}} = T_{\text{B}}]$ is simulated by the dotted line in Fig.(4). Yet, the solid line models the Tsallis distribution for a broader range of examined charmed quark energies. It appears as a deviation from the Boltzmann distribution. Boltzmann distribution is sufficient when the ratio of the transport coefficients specified by the right-hand side of eq(32) is constant. However, when the ratio is linear in E, the statistics are described by Tsallis distribution. Fig.(4) also provides Tsallis statistics parametrization values $[q = 1.08, T_{\text{T}} = 123.6 MeV]$ according to the linear regression fit.[9], [11] The values of transport coefficients can be calculated from [12]

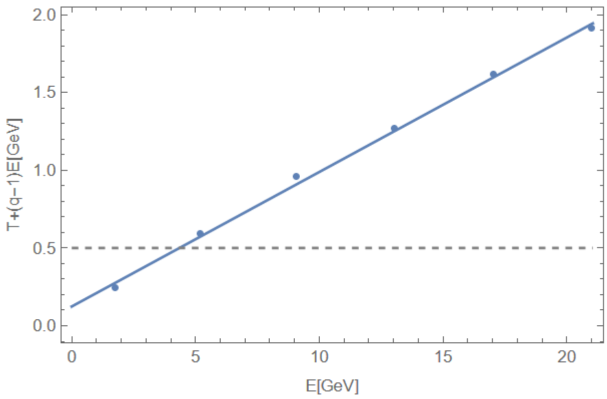

Figure 4: Fig.(4) Linear regression for the coefficients ratio of eq(32) for a $1.5 GeV$ charm quark thermalized at $T = 0.5 GeV$ in gluon medium

5 Conclusion

The study of Quark-gluon plasma (QGP) is an access to the micro-second old universe. It is a state of matter that can be produced in high-energy heavy-ion collisions. Heavy-ion collisions at Large Hadron Collider (LHC) at CERN produce a large number of heavy quarks. Studying heavy quarks in QGP provides insights into the behavior of matter at very high temperatures and densities. The equilibrium of HQs inside QGP is affected by the temperature and density of the QGP, as well as the transport properties of HQs, their masses, and their momenta. Through the transport equations; BTE and FPE, we have demonstrated that the shape of the equilibrium distribution is formed by the ratio between drag and diffusion coefficients. Finally, with the help of the calculated values of the transport coefficients, we have demonstrated that the thermalization of charm quarks by collisional processes with gluons results in a spectral configuration well parameterized by the Tsallis distribution with two parameters q and T.

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